Problem #1 Fall 2003

$$H = J\left(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$$\vec{S}_1 \cdot \vec{S}_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$$

$$H = J\left(\vec{S}_1\cdot\vec{S}_2 + (K-1) S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^2 + 2\vec{S}_1\vec{S}_2 + \vec{S}_2^2$$

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - S_2^2\right)$$

$$H = J\left(\frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) + (K-1) S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$\frac{1}{2}(\frac{1}{2}+1)$

(Sym) $1(1+1)$
(Asym) $0(0+1)$

a) antisymmetric spatial wave function ...

For fermions the total wave function must be anti-symmetric, therefore the spin piece must be symmetric

$\langle \text{sym} | H | \text{sym} \rangle$

$$\langle H \rangle_{sym} = \langle 1\, m | H | 1\, m \rangle$$

$$= J\left(\frac{\hbar^2}{2} \left\{ \underbrace{1(1+1)}_{2} - \underbrace{\underbrace{2\cdot\frac{1}{2}\left(\frac{1}{2}+1\right)}_{\frac{3}{4}}}_{\frac{3}{2}} \right\} + (K-1)\hbar^2 m_1 m_2 \right) + \mu\hbar(m_1+m_2)B$$

$$\boxed{E = J\hbar^2\left(\frac{1}{4} + (K-1)m_1 m_2\right) + \mu B\hbar(m_1+m_2)}$$

$$E_{\uparrow\uparrow} = \langle \uparrow\uparrow | H | \uparrow\uparrow \rangle = J\hbar^2\left(\frac{1}{4} + \frac{(K-1)}{4}\right) + \mu B\hbar = \boxed{\frac{J\hbar^2 K}{4} + \mu B\hbar}$$

$$E_{\downarrow\downarrow} = \langle \downarrow\downarrow | H | \downarrow\downarrow \rangle = J\hbar^2\left(\frac{1}{4} + \frac{(K-1)}{4}\right) - \mu B\hbar = \boxed{\frac{J\hbar^2 K}{4} - \mu B\hbar}$$

$$E_{\downarrow\uparrow} = \frac{1}{2}\left[\langle \uparrow\downarrow | H | \uparrow\downarrow \rangle + \langle \downarrow\uparrow | H | \downarrow\uparrow \rangle\right] = \frac{2J\hbar^2}{2}\left(\frac{1}{4} - \frac{(K-1)}{4}\right)$$

$$E_{\downarrow\uparrow} = J\hbar^2\left(\frac{1}{4} - \frac{(K-1)}{4}\right) = \boxed{\frac{J\hbar^2(2-K)}{4}}$$

b) for antisymmetric state

$$E = J\hbar^2\left(-\frac{3}{4} + (K-1)m_1 m_2\right) + \mu\hbar B(\underbrace{m_1+m_2}_{\to 0})$$

$$E_{\uparrow\downarrow} = \frac{1}{2}\left[\langle \uparrow\downarrow | H | \uparrow\downarrow \rangle - \langle \downarrow\uparrow | H | \downarrow\uparrow \rangle\right] = J\hbar^2\left(-\frac{3}{4} - \frac{(K-1)}{4}\right)$$

$$= \boxed{J\hbar^2\left(-\frac{(K-2)}{4}\right)}$$