Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J\left(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z \right) + \mu\left(S_1^z + S_2^z\right) B$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wave function.

The total wave function for electrons must be antisymmetric, so, if the spatial part is anti-symmetric, then the spin part must be symmetric to preserve the symmetry.

Now, let's rewrite the Hamiltonian into a more user friendly form.

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2 = S_1^2 + S_2^2 + 2\left(S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z\right)$$

$$\Rightarrow \quad S_1^x S_2^x + S_1^y S_2^y = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) - S_1^z S_2^z$$

So, then the Hamiltonian becomes

$$H = J\left[\frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) + (K-1) S_1^z S_2^z \right] + \mu B\left(S_1^z + S_2^z\right)$$

note $S_i^2 = S_i(S_i + 1) = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}$

So,

$$H = J\left[\frac{1}{2} S^2 - \frac{3}{4} + (K-1) S_1^z S_2^z \right] + \mu B\left(S_1^z + S_2^z\right)$$

what are the possible values of S?

$$|S_1 - S_2| \le S \le |S_1 + S_2|$$

$$\left|\frac{1}{2} - \frac{1}{2}\right| \le S \le \left|\frac{1}{2} + \frac{1}{2}\right|$$

$$0 \le S \le 1$$

$$\Rightarrow S = 0 \text{ or } 1$$

For $S = 0$, $m_s = 0$ and the only state possible is

$$|00\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad (1)$$

note: $|00\rangle$ is anti-symmetric ... if you let $\uparrow \to \downarrow$ and $\downarrow \to \uparrow$, you get back $-|00\rangle$

For $s=1$, $m_s = -1, 0, 1$ and the 3 states are

$$|1,-1\rangle = |\downarrow\downarrow\rangle$$

$$|1,0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad (2)$$

$$|1,1\rangle = |\uparrow\uparrow\rangle$$

note: all these states are symmetric. let $\uparrow \to \downarrow$ and $\downarrow \to \uparrow$ in $|1,0\rangle$ and you get back $|1,0\rangle$ ... The other two are trivially symmetric. So, for part (c), we want to find the energy levels of the $S=1$ (symmetric spin part).

So, we have

$$\underline{|1,-1\rangle} \quad \langle 1,-1|H|1,-1\rangle = J\left[\tfrac{1}{2}S(S+1) - \tfrac{3}{4} + (K-1)S_1^z S_2^z\right] + \mu B(S_1^z + S_2^z)\Big|_{S=1}$$

$$= J\left[1 - \tfrac{3}{4} + (K-1)\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right] + \mu B\left(-\tfrac{1}{2} + -\tfrac{1}{2}\right)$$

$$= J\left[\tfrac{1}{4} + \tfrac{K}{4} - \tfrac{1}{4}\right] - \mu B$$

$$\therefore \boxed{E_{1,-1} = \frac{JK}{4} - \mu \vec{B}}$$

$|1,0\rangle$ $\langle 10|H|10\rangle = \frac{1}{2}\Big[J\left(\frac{1}{2}\cdot 2 - \frac{3}{4} + (K-1)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right) + \mu B\left(\frac{1}{2}-\frac{1}{2}\right) +$

$+ J\left(1 - \frac{3}{4} + (K-1)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right) + \mu B\left(-\frac{1}{2}+\frac{1}{2}\right)\Big]$

$= \frac{1}{2}\left[J\left(\frac{1}{4} - \frac{K}{4} + \frac{1}{4}\right) + J\left(\frac{1}{4} - \frac{K}{4} + \frac{1}{4}\right)\right]$

$= \frac{2J}{2}\left(\frac{1}{2} - \frac{K}{4}\right)$

$\therefore \boxed{E_{10} = J\left(\frac{1}{2} - \frac{K}{4}\right)}$

$|1,1\rangle$ $\langle 1,1|H|1,1\rangle = J\left[\frac{1}{4} + (K-1)\left(\frac{1}{4}\right)\right] + \mu B\left(\frac{1}{2}+\frac{1}{2}\right)]$

$= J\left(\frac{K}{4}\right) + \mu B$

$\Rightarrow \boxed{E_{11} = \frac{JK}{4} + \mu B}$

b) repeat for a symmetric spatial wave function

Now, we use the anti-symmetric spin part (eq (1)).

$|0,0\rangle$ $\langle 00|H|00\rangle = \frac{J}{2}\left[-\frac{3}{4} + (K-1)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right] + \frac{\mu B}{2}\cancelto{0}{\left(\frac{1}{2}-\frac{1}{2}\right)}$

$-\frac{J}{2}\left[-\frac{3}{4} + (K-1)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right] - \frac{\mu B}{2}\cancelto{0}{\left(-\frac{1}{2}+\frac{1}{2}\right)}$

$\Rightarrow \boxed{E_{00} = 0}$