QM F'03 #5

H-atom is in the ground state ($|100\rangle$) at $t<0$.

A time dependent electric field is applied

$$\mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \quad \text{for } t>0$$

A long time passes.

a) what is the fraction of atoms in the $|200\rangle$ state?

$$C_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle 100|H'|200\rangle e^{\frac{i\omega_0 t}{\hbar}}\,dt$$

$$\langle 100|H'|200\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{2}{a_0^{3/2}} e^{-r/a_0}\frac{1}{\sqrt{4\pi}}\left(e\mathcal{E}_0 \overset{r\cos\theta}{\overset{\Downarrow}{z}} e^{-t/\tau}\right)\frac{1}{\sqrt{2}}\frac{1}{a_0^{3/2}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\frac{1}{\sqrt{4\pi}}\,r^2 dr\sin\theta\, d\theta\, d\phi$$

$$= \frac{2e}{4\pi a_0^3}\frac{\mathcal{E}_0}{\sqrt{2}}e^{-t/\tau}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}\underbrace{\int_0^{\pi}\cos\theta\sin\theta\, d\theta}\int_0^{\infty} r^3\left(1-\frac{r}{2a_0}\right)e^{-\frac{3r}{2a_0}}dr = 0$$

$u=\sin\theta$

$du=\cos\theta\, d\theta$

$$\int_0^0 u\,du = 0$$

So $\langle 100|H'|200\rangle = 0$, hence there will be 0 atoms in the 2p state.

b) what is the fraction of atoms in the 2p state?

2p: $|211\rangle, |21{-1}\rangle, |210\rangle$

$$C_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle 100|H'|21m\rangle e^{i\omega_0 t}\,dt$$

$\cdot\ \langle 100|H'|211\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{2}{a_0^{3/2}} \frac{e^{-r/a_0}}{\sqrt{4\pi}} \left(e\mathcal{E}_0\, r\cos\theta\, e^{-t/\tau}\right) \frac{1}{\sqrt{24}} \frac{1}{a_0^{5/2}}\, r\, e^{-\frac{r}{2a_0}} - \left(\frac{+3}{8\pi}\right)^{1/2} \sin\theta\, e^{i\phi}\, r^2 dr\, \sin\theta\, d\theta\, d\phi$

$$= \frac{-2e\mathcal{E}_0 e^{-t/\tau}}{a_0^4 \sqrt{4\pi}\sqrt{24}} \left(\frac{3}{8\pi}\right)^{1/2} \underbrace{\int_0^{2\pi} e^{i\phi} d\phi}_{} \int_0^{\pi} \cos\theta\, \sin^2\theta\, d\theta \int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}} dr = 0$$

$$\frac{1}{i} e^{i\phi}\Big|_0^{2\pi} = \frac{1}{i}\left[1 - 1\right] = 0$$

So $\langle 100|H'|211\rangle = 0$, similarly $\langle 100|H'|21{-1}\rangle$ as $Y_{1,-1} = (-1)Y_{1,1}^*$

$\cdot\ \langle 100|H'|210\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{2e^{-r/a_0}}{a_0^{3/2}\sqrt{4\pi}} \left(e\mathcal{E}_0\, r\cos\theta\, e^{-t/\tau}\right) \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \;\; \frac{1}{\sqrt{24}}\left(\frac{1}{a_0}\right)^{5/2} r\, e^{-r/2a_0} \; r^2 dr\, \sin\theta\, d\theta\, d\phi$

$$= \frac{2e\mathcal{E}_0 e^{-t/\tau}}{4\pi a_0^4} \frac{1}{\sqrt{8}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \cos^2\theta\, \sin\theta\, d\theta}_{} \underbrace{\int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}} dr}_{}$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

$$-\int_1^{-1} u^2 du = +\int_{-1}^{1} u^2 du = \frac{1}{3}\left[1+1\right] = 2/3$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}\ ;\quad n = 4;\ a = \frac{3}{2a_0}$$

$$= \frac{4!}{\left(\frac{3}{2a_0}\right)^5}$$

$$= \frac{2e\mathcal{E}_0 e^{-t/\tau}}{4\pi\sqrt{8}\, a_0^4}\, 2\pi \cdot \frac{2}{3} \cdot \frac{4!}{\left(\frac{3}{2a_0}\right)^5} = \frac{16\, e\mathcal{E}_0}{\sqrt{8}} e^{-t/\tau} a_0 \frac{2^5}{3^5}$$

So

$$c_{1\to2}(t) = \frac{+i}{\hbar} \frac{16 e\mathcal{E}_0 a_0}{\sqrt{8}} \left(\frac{2}{3}\right)^5 \int_0^{\infty} e^{-t/\tau} e^{i\omega_0 t} dt = \frac{+i}{\hbar} \frac{16 e\mathcal{E}_0 a_0}{\sqrt{8}} \left(\frac{2}{3}\right)^5 \int_0^{\infty} e^{-(1/\tau - i\omega_0)t} dt$$

$$= \frac{i}{\hbar} \frac{16 e\mathcal{E}_0 a_0}{\sqrt{8}} \left(\frac{2}{3}\right)^5 \left(\frac{-1}{\frac{1}{\tau} - i\omega_0}\right) e^{-(1/\tau - i\omega_0)t}\Big|_0^{\infty} = \frac{i}{\hbar} \frac{16 e\mathcal{E}_0 a_0}{\sqrt{8}} \left(\frac{2}{3}\right)^5 \frac{1}{(1/\tau - i\omega_0)}$$

The probability is given by: $|c_{1\to2}(t)|^2 = \frac{16^2 e^2 \mathcal{E}_0^2 a_0^2}{8\hbar^2} \left(\frac{2}{3}\right)^{10} \frac{1}{(1/\tau - i\omega_0)^2} = \frac{32 e^2 \mathcal{E}_0^2 a_0^2}{\hbar^2} \left(\frac{2}{3}\right)^{10} \frac{1}{1/\tau^2 + \omega_0^2}$

$\uparrow\ \frac{1}{\frac{1}{\tau^2} + \omega_0^2}$

And hence if the total population of atoms is N the the fraction of atoms in the 2p state is:

$$\left(\frac{32 e^2 \mathcal{E}_0^2 a_0^2}{\hbar^2} \left(\frac{2}{3}\right)^{10} \frac{1}{1/\tau^2 + \omega_0^2}\right) N$$