QM S'03 #1

$V(x) = \frac{1}{2} m\omega^2 (x - \epsilon(t))^2$; $\epsilon(t) = \epsilon\, e^{-t^2/\tau^2}$ $\quad \epsilon \ll 1$

At $t = -\infty$ the particle is in the ground state ($n=0$) what are the possible states the particle can be in at $t = \infty$. Work to lowest order of ϵ.

$$V(x) = \frac{1}{2} m\omega^2 (x^2 - 2x\epsilon(t) + \epsilon^2(t)) = \frac{1}{2} m\omega^2 x^2 - m\omega^2 x\,\epsilon(t) + \underbrace{\cancel{\tfrac{1}{2} m\omega^2 \epsilon^2(t)}}_{\text{disregard}}$$

$$c(t) = \frac{-i}{\hbar} \int_{-\infty}^{\infty} \langle H' \rangle e^{i\omega_0 t} \quad ; \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Now $H' = -m\omega^2 x\, \epsilon\, e^{-t^2/\tau^2}$ and $\langle H' \rangle = \langle n | H' | 0 \rangle$

$$\Rightarrow \; -m\omega^2 \epsilon\, e^{-t^2/\tau^2} \langle n | X | 0 \rangle \quad ; \quad \text{but} \quad X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$\Rightarrow -m\omega^2 \epsilon\, e^{-t^2/\tau^2} \left(\sqrt{\frac{\hbar}{2m\omega}} \left[\cancel{\langle n | a | 0 \rangle}^{\,0} + \langle n | a^\dagger | 0 \rangle \right] \right)$$

$= \sqrt{1+0}$ for $n=1$
$= 0$ for $n \neq 1$

$$= -m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}}\, e^{-t^2/\tau^2} \quad \text{and} \quad \omega_0 = \frac{E_1 - E_0}{\hbar} = \frac{(1+\frac{1}{2})\hbar\omega - (0+\frac{1}{2})\hbar\omega}{\hbar} = \frac{3}{2}\omega - \frac{1}{2}\omega = \underline{\omega}$$

$$\text{so} \quad c_{0\to 1}(t) = \frac{-i}{\hbar} \left(-m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}} \right) \int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega t}\, dt$$

we need to complete the square:

$$\int_{-\infty}^{\infty} e^{-(t^2/\tau^2 - i\omega t)}\, dt \qquad y \equiv \sqrt{a}\, t + \frac{b}{2\sqrt{a}} \Rightarrow y^2 = a t^2 + b t + \frac{b^2}{4a}$$

$$\text{so} \quad a = \frac{1}{\tau^2} \; ; \; b = -i\omega \; \Rightarrow \; y = \frac{1}{\tau} t - \frac{i\omega\tau}{2} \qquad dy = \frac{1}{\tau}\, dt$$

so $\displaystyle\int_{-\infty}^{\infty} e^{-(t^2\tau^{-2} - i\omega t)}\,dt = \frac{1}{\tau^{-1}}\int_{-\infty}^{\infty} e^{-(y^2 - \frac{b^2}{4a})}\,dy = \frac{e^{b^2/4a}}{\tau^{-1}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{2\int_0^{\infty} e^{-y^2}dy = \sqrt{\pi}} = \frac{\sqrt{\pi}}{\tau^{-1}}\, e^{b^2/4a}$

hence $\displaystyle c_{0\to1}(t) = \frac{i\, m\, \omega^2 \epsilon}{\hbar\, \tau^{-1}} \sqrt{\frac{\hbar \pi}{2 m \omega}}\; e^{b^2/4a}$

and the probability is: $\displaystyle |c_{0\to1}|^2 = \frac{m^{\cancel{2}\,1}\omega^{\cancel{4}\,3}\epsilon^2}{\hbar^{\cancel{2}\,1}\tau^{-2}}\; \frac{\cancel{\hbar}\pi}{2\cancel{\hbar}\cancel{\omega}}\; e^{b^2/2a}$

$$= \frac{m\omega^3\epsilon^2\pi}{2\hbar\tau^{-2}}\, e^{(-i\omega)^2/2(\frac{1}{\tau^2})} = \frac{m\omega^3\epsilon^2\pi}{2\hbar\tau^{-2}}\, e^{-\frac{\omega^2\tau^2}{2}} = \boxed{\frac{m\omega^3\epsilon^2\tau^2\pi}{2\hbar}\, e^{-\frac{\omega^2\tau^2}{2}}}$$