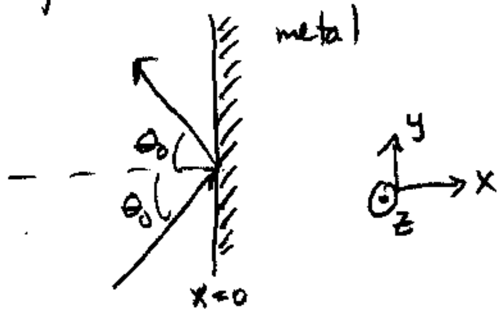
Spring 2003 # 10 (p 1 of 2)

X-ray Mirror: X-rays which strike the metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. As shown below, the metal occupies the region x>0. The x-rays propagate in the x-y plane and their polarization is in the z-direction, coming out of the page. Assume that the metal contains n free electrons per unit volume and is non-magnetic. Derive an expression for the critical angle θ_0.

(see Lim Yung-Kuo #4029 part(a))



Snell's law is applicaple to this poblem (note: snell's law is independent of frequency). That is,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

for critical angle, we have $\theta_1 = \theta_0$ & $\theta_2 = 90°$. Assume $n_1 = 1$. So, we have

$$\sin\theta_0 = n_2$$

$$\Rightarrow \quad \theta_0 = \sin^{-1}(n_2) \qquad (1)$$

What is n_2?

The equation of motion for an electron in a field of X-rays is (from Lorentz force law)

$$m\ddot{x} = -eE_0 e^{-i\omega t} = -eE \qquad (2)$$

the solution to this D.E. is

$$x = x_0 e^{-i\omega t} \quad \Rightarrow \quad \ddot{x} = -\omega^2 x$$

substituting this result into eq (2) yields

$$m\omega^2 x = eE \qquad (3)$$

← see Jackson 2nd ed Section 4.6 (Molecular Polarizabilit

The dipole moment is given by

$$p = -ex \overset{\text{eq (3)}}{=} \frac{-e^2}{m\omega^2} E$$

Then the induced polarization is

$$\underline{P} = -\frac{ne^2}{m\omega^2} E = \chi_e E \qquad \begin{pmatrix}\text{Jackson eq 4.36} \\ \text{2nd ed}\end{pmatrix}$$

where n is the density of electrons.

So,

$$\chi_e = \frac{-ne^2}{m\omega^2}$$

note! from Jackson eq 10.79 2nd ed. $\omega_p^2 = \frac{ne^2}{m}$

Then,

$$\chi_e = -\frac{\omega_p^2}{\omega^2} \qquad (4)$$

And finally the index refraction of a metal is given by

$$n = \sqrt{1 + \chi_e}$$

So,

$$n_2 = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad (5)$$

Substituting eq (5) into eq (1) yields the expression for the critical angle

$$\boxed{\theta_0 = \sin^{-1}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}}$$