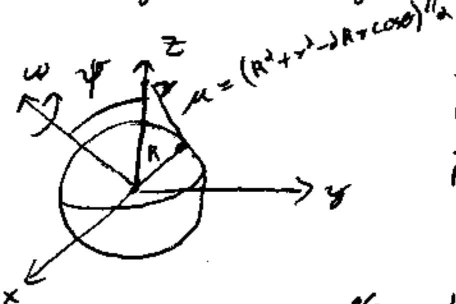
S'03 E.M. #12

a) Show that the field inside of a sphere of uniformly magnetized material ($\vec{M} = M\hat{z}$) is:

$$\vec{B} = \frac{2}{3}\mu_0 M\hat{z}$$

Two ways of doing it:



$$\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$$
$$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} = M\hat{\phi}$$

Now K_b can be thought of as being due to a surface charge σ on a rotating (ω) sphere.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\rho}\,da' = \frac{\mu_0\sigma}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\vec{v}(\vec{r}')\,R^2\sin\theta\,d\theta\,d\phi}{\sqrt{R^2+r^2-2Rr\cos\theta}}$$

now $\vec{v}(\vec{r}') = \vec{\omega}\times\vec{r}'$; $\vec{r}' = R\sin\theta\cos\phi\,\hat{x} + R\sin\theta\sin\phi\,\hat{y} + R\cos\theta\,\hat{\phi}$

$$\vec{\omega} = \omega\sin\psi\,\hat{x} + 0\,\hat{y} + \omega\cos\psi\,\hat{z}$$

hence; $$\vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta\cos\phi & R\sin\theta\sin\phi & R\cos\theta \end{vmatrix}$$

$= \hat{x}(-R\omega\sin\theta\cos\phi\cos\psi) - \hat{y}(R\omega\cos\theta\sin\psi - R\omega\sin\theta\cos\phi\cos\psi)$
$+ \hat{z}(R\omega\sin\theta\sin\phi\sin\psi)$

now $\int_0^{2\pi}\sin\phi\,d\phi = 0 = \int_0^{2\pi}\cos\phi\,d\phi$, hence all of the above terms with either $\sin\phi$ or $\cos\phi$ in them will vanish: we are left with

$$\vec{\omega}\times\vec{r}' = -R\omega\cos\theta\sin\psi\,\hat{y}$$

so

$$\vec{A} = -\frac{\mu_0 \sigma R^3 \omega \sin\psi}{4\pi} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}\, \hat{y}$$

$$= \underbrace{-\frac{\mu_0\sigma R^3\omega\sin\psi}{2}}_{\equiv C} \int_0^{\pi} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}\, \hat{y}$$

let $u=\cos\theta \Rightarrow du = -\sin\theta\, d\theta$

$$= C\int_1^{-1} \frac{u\,du}{\sqrt{R^2+r^2-2Rru}}\, \hat{y}$$

This integral is just

$$\int \frac{x\,dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2}(ax+b)^{1/2}$$

with $a=-2Rr$; $b=R^2+r^2$

$$= C\left[\frac{2(-2Rru-2(R^2+r^2))}{3(4R^2r^2)}(-2Rru+R^2+r^2)^{1/2}\right]_{+1}^{-1}\hat{y}$$

$$= C\left[-\frac{(R^2+r^2+Rru)}{3R^2r^2}(R^2+r^2-2Rru)^{1/2}\right]_{+1}^{-1}$$

at the cost of a minus sign we can swith the limits

$$= +\frac{C}{3R^2r^2}\left[(R^2+r^2+Rr)\underbrace{(R^2+r^2-2Rr)^{1/2}}_{\substack{R-r \quad r<R\\ r-R \quad r>R}} - (R^2+r^2-Rr)\underbrace{(R^2+r^2+2Rr)^{1/2}}_{(R+r)}\right]$$

we are interrested in the case $r<R$:

$$= +\frac{C}{3R^2r^2}\left[(R^2+r^2+Rr)(R-r) - (R^2+r^2-Rr)(R+r)\right]$$

$$= +\frac{C}{3R^2r^2}\left[\cancel{R^3} + \cancel{Rr^2} + \cancel{R^2r} - \cancel{R^2r} - r^3 - \cancel{Rr^2} - \cancel{R^3} - \cancel{Rr^2} + \cancel{R^2r} - \cancel{R^2r} - r^3 + \cancel{Rr^2}\right]$$

$$= +\frac{C}{3R^2r^2}\left[-2r^3\right] = -\frac{2}{3}\frac{C}{R^2}\, r\, \hat{y}$$

or as $-r\omega\sin\psi = \vec{\omega}\times\vec{r}$

So $\vec{A}(\vec{r}) = -\frac{\cancel{2}}{3}\frac{\mu_0\sigma R^{\cancel{3}\,1}\omega\sin\psi\, r}{\cancel{2}\cancel{R^2}}\,\hat{y} = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$

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Now aligning $\vec{\omega}$ to coincide with the $\hat{z}$ axis:

$$\vec{A}(\vec{r},\vec{\theta},\vec{\phi}) = \underbrace{\frac{\mu_0 R\omega\sigma}{3}}_{c'} r\sin\theta\hat{\phi}$$

To get $\vec{B}$:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta A_\phi - \frac{\partial}{\partial\phi}\cancelto{0}{A_\theta}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cancelto{0}{A_r} - \frac{\partial}{\partial r} r A_\phi\right]\hat{\theta}$$

$$= \frac{1}{\cancel{r}\sin\theta} c'\cancel{r}\frac{\partial}{\partial\theta}\sin^2\theta\ \hat{r} + -\frac{c'\sin\theta}{r}\frac{\partial}{\partial r} r^2\hat{\theta}$$

$$= c'\left[\frac{1}{\cancel{\sin\theta}} 2\cancel{\sin\theta}\cos\theta\hat{r} - \frac{\sin\theta}{\cancel{r}} 2\cancel{r}\hat{\theta}\right]$$

$$= 2c'\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$

now $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$

$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$

so

$$\left.\begin{aligned} \cos\theta\hat{r} &= \cancel{\cos\theta\sin\theta\cos\phi\hat{x}} + \cancel{\cos\theta\sin\theta\sin\phi\hat{y}} + \cos^2\theta\hat{z} \\ -\sin\theta\hat{\theta} &= \cancel{-\cos\theta\sin\theta\cos\phi\hat{x}} \cancel{- \cos\theta\sin\theta\sin\phi\hat{y}} + \sin^2\theta\hat{z} \end{aligned}\right\}\ \hat{z}$$

So $$\boxed{\vec{B} = 2c'\hat{z} = \frac{2\mu_0}{3}\underbrace{R\omega\sigma}_{=k_b=M}\hat{z} = \frac{2}{3}\mu_0 M\hat{z}}$$

The second method is via boundary conditions:

As $\vec{\nabla} \times \vec{H} = 0$, this means $\vec{H} = -\vec{\nabla W}$ just like in the electrical case. So one B.C. is

$$W_{above} = W_{below}$$

the second B.C. comes from

$$H_{above}^{\perp} - H_{below}^{\perp} = -(M_1^{\perp} - M_2^{\perp})$$

$M_{above} = 0$ $\quad M_{below} = M\cos\theta$

which becomes $$\frac{\partial W_{above}}{\partial r} - \frac{\partial W_{below}}{\partial r} = \vec{M}\cdot\hat{n} = (M_1^{\perp} - M_2^{\perp})$$

now we can use the usual Legendre polynomial stuff:

$$W_{above} = \sum_{\ell} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta); \quad W_{below} = \sum_{\ell} A_\ell r^\ell P_\ell(\cos\theta)$$

now from $\quad W_{above} = W_{below} \Rightarrow \dfrac{B_\ell}{r^{\ell+1}} = A_\ell r^\ell \Rightarrow B_\ell = A_\ell r^{2\ell+1}$

from the second B.C.

$$-(\ell+1) \sum_{\ell} \frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) - \sum_{\ell} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -M\cos\theta$$

only works for $\ell = 1$, so

$$2\frac{B_1}{R^3} + A_1 = M \qquad \text{but} \quad B_1 = A_1 R^3$$

so $\qquad 2A_1 + A_1 = M \Rightarrow A_1 = \dfrac{M}{3} \Rightarrow B_1 = \dfrac{M}{3} R^3$

hence $\qquad W_{\substack{inside \\ (below)}} = \dfrac{M}{3} r\cos\theta$

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and

$$\vec{H} = -\vec{\nabla} W = -\left[\frac{\partial}{\partial r} W \hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta} W \hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} W \hat{\phi} \right]$$

$$= -\left[\frac{M}{3}\cos\theta\hat{r} - \frac{M}{3}\sin\theta\hat{\theta} + 0 \right]$$

now $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$

hence

$$\left.\begin{aligned} \cos\theta\hat{r} &= \cancel{\cos\theta\sin\theta\cos\phi\hat{x}} + \cancel{\cos\theta\sin\theta\sin\phi\hat{y}} + \cos^2\theta\hat{z} \\ -\sin\theta\hat{\theta} &= \cancel{-\cos\theta\sin\theta\cos\phi\hat{x}} \cancel{- \cos\theta\sin\theta\sin\phi\hat{y}} + \sin^2\theta\hat{z} \end{aligned}\right\} = \hat{z}$$

so $\vec{H} = -\frac{M}{3}\hat{z}$

but $\vec{H} = \frac{1}{\mu_0}\vec{B} - \underbrace{\vec{M}}_{M\hat{z}} \Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M})$

$$= \mu_0\left(-\frac{M}{3} + M\right)\hat{z}$$

$$= \boxed{\frac{2}{3}\mu_0 M\hat{z}}$$

b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_0\hat{z}$. Using the above result, find the magnetic field inside the sphere.

$$\vec{M} = \chi_m\vec{H} = \chi_m\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\overbrace{\left[\vec{B}_0 + \frac{2}{3}\mu_0\vec{M}\right]}^{=\vec{B}} - \vec{M}\right)$$

$$= \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M} - \vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

so $\vec{M}\left(1 + \frac{\chi_m}{3}\right) = \frac{\chi_m}{\mu_0}\vec{B}_0 \Rightarrow \vec{M} = \frac{\chi_m/\mu_0\,\vec{B}_0}{\left(1+\frac{\chi_m}{3}\right)}$

Now $\vec{B} = \vec{B}_0 + \vec{B}_{sphere} = \vec{B}_0 + \frac{2}{3}\mu_0\frac{\chi_m/\mu_0}{(1+\chi_m/3)}B_0 = \left(1 + \frac{2\chi_m}{(3+\chi_m)}\right)\vec{B}_0 = \boxed{\left(\frac{1+\chi_m}{1+\chi_m/3}\right)\vec{B}_0}$