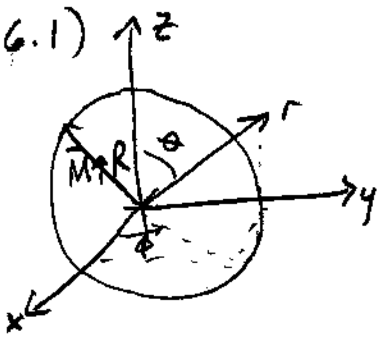
(a) Show that the field inside a sphere of uniformly magnetized material ($\vec{M} = M\hat{z}$) is

$$\vec{B} = \frac{2}{3}\mu_0 M \hat{z}$$

(see Griffiths' example 6.1)



the current density is given by

$$j_m = \nabla \times \vec{m} = 0$$

and the surface current is

$$|\vec{K}| = |\vec{M} \times \hat{n}| = M|\hat{z} \times \hat{r}| = M\sin\theta$$

note that a rotating spherical shell of uniform surface charge has

$$K = \sigma v = \sigma \omega R \sin\theta$$

So, the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell with $M \to \sigma R \omega$. So, let's find the field of a spinning spherical shell (see Griffiths' example 5.11). From Griffiths' example 5.11, we have that the vector potential is given by (eq 5.67)

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r\sin\theta\,\hat{\phi} & r < R \\ \dfrac{\mu_0 R^4 \omega \sigma}{3r^2}\sin\theta\,\hat{\phi} & r > R \end{cases}$$

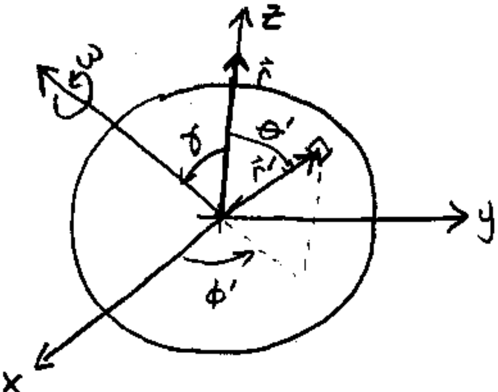
How are these derived? see Jackson problem 5.13. See next 8 pages for 2 different ways to do this.

Jackson 5.13 A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and the magnetic-flux density both inside and outside the sphere.

From Griffiths' eqn. 5.64, we know that the vector potential can be written in terms of the surface current as follows

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,da'$$

taking the advice in Griffiths' example 5.11, let's orient the coordinates such that $\vec{r}$ lies along the z-axis and the axis of rotation lies in the xz plane. That is, (Griffiths' Figure 5.46)



(i) First, let's consider what $\vec{K}(\vec{r}\,')$ is. From Griffiths' eqn 5.23, we know

$$\vec{K} = \sigma\vec{v}$$

where $\vec{v} = \vec{\omega}\times\vec{r}\,'$ is the velocity of a point $\vec{r}\,'$ in a rotating rigid body (see Griffiths' example 5.11). Now, since we oriented the coordinate system such that the axis of rotation is in the xz plane, we have

$$\vec{\omega} = \omega\sin\gamma\,\hat{x} + \omega\cos\gamma\,\hat{z}$$

we also know that in spherical coordinates, $\vec{r}\,'$ is given by

$$\vec{r}\,' = r\sin\theta'\cos\phi'\,\hat{x} + r\sin\theta'\sin\phi'\,\hat{y} + r\cos\theta'\,\hat{z}$$, where $r=a$ for our case

so,

$$\vec{K} = \sigma(\vec{\omega}\times\vec{r}\,') = \sigma\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\gamma & 0 & \omega\cos\gamma \\ a\sin\theta'\cos\phi' & a\sin\theta'\sin\phi' & a\cos\theta' \end{vmatrix}$$

$$= \sigma\Big[\hat{x}(-\omega a\sin\theta'\sin\phi'\cos\gamma) + \hat{y}(\omega a\sin\theta'\cos\phi'\cos\gamma - \omega a\sin\gamma\cos\theta') + \hat{z}(a\omega\sin\gamma\sin\theta'\sin\phi')\Big]$$

(ii) Now, note

$$|\vec{r}-\vec{r}'| = \sqrt{|\vec{r}-\vec{r}'|^2} = \left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{1/2} = \left[r^2+(r')^2-2rr'\cos\theta'\right]^{1/2}\Big|_{r'=a}$$

so, we have

$$|\vec{r}-\vec{r}'| = \left[r^2+a^2-2ra\cos\theta'\right]^{1/2}$$

(iii) now consider da'

For our case, da' is given by

$$da' = a^2 \sin\theta'\, d\theta' d\phi'$$

Putting the results from parts (i), (ii), and (iii) into our expression for $\vec{A}(\vec{r})$, we get

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\, d\theta' \left[\frac{-\sin\theta'\sin\phi'\cos\gamma\,\hat{x} + (\sin\theta'\cos\phi'\cos\gamma - \sin\gamma\cos\theta')\hat{y} + \sin\gamma\sin\theta'\sin\phi'\,\hat{z}}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}\right]$$

Now, consider the integration over ϕ'. Note that

$$\int_0^{2\pi} d\phi' \sin\phi' = -\left[\cos\phi'\right]_0^{2\pi} = -\left[1-1\right] = 0$$

and

$$\int_0^{2\pi} d\phi' \cos\phi' = \left[\sin\phi'\right]_0^{2\pi} = 0-0 = 0$$

with this in mind, the integral over the $\hat{x}$ and $\hat{z}$ directions vanish as well as the first term in the $\hat{y}$ direction. So, the only non-zero term that survives is

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\, d\theta' \left[\frac{-\sin\gamma\cos\theta'}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}\right]$$

$$= \frac{2\pi\sigma\omega a^3}{c}\sin\gamma\int_0^{\pi} d\theta' \frac{(-\sin\theta'\cos\theta')}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}$$

In order to solve this integral, we need to make the following substitution

$$u = \cos\theta' \Rightarrow du = -\sin\theta'\, d\theta'$$

So, we have

$$\vec{A}(\vec{r}) = \frac{2\pi\sigma\omega a^3}{c}\sin\gamma \int_1^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}}$$

note: from Schaum's Outlines Mathematical Handbook of Formulas and Tables, eqn. 17.2.2, we know

$$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

for our case,
$x \to u$
$b \to r^2+a^2$
$a \to -2ra$

So, we have

$$\int_{+1}^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}} = \left[2\frac{\left[(-2ra)u-2r^2-2a^2\right]}{3(-2ra)^2}\sqrt{-2rau+r^2+a^2}\right]_1^{-1}$$

$$= \frac{2}{12r^2a^2}\left[2ra-2r^2-2a^2\sqrt{2ra+r^2+a^2} - \left(-2ra-2r^2-2a^2\right)\sqrt{-2ra+r^2+a^2}\right]$$

$$= \frac{4}{12r^2a^2}\left[-\left(r^2+a^2-ra\right)\sqrt{(r+a)^2} + \left(r^2+a^2+ra\right)\sqrt{(r-a)^2}\right]$$

$$= \frac{1}{3r^2a^2}\left[-\left(r^2+a^2-ra\right)(r+a) + \left(r^2+a^2+ra\right)|r-a|\right]$$

here we have left $r-a$ in absolute value signs because the result of $r-a$ must remain positive in order for the anser to be real. This is the point where the solution will vary if $r<a$ or $r>a$. So, if

(i) $r<a$

$$\int_{+1}^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}} = \frac{1}{3r^2a^2}\left[-\left(r^2+a^2-ra\right)(r+a) + \left(r^2+a^2+ra\right)(a-r)\right]$$

$$= \frac{1}{3r^2a^2}\Big[-\left(r^3+ra^2-r^2a+r^2a+a^3-ra^2\right) +$$

$$+\; r^2a+a^3+ra^2-r^3-ra^2-r^2a\Big]$$

$$\Rightarrow \int_{+1}^{-1} \frac{u\,du}{[r^2+a^2-2rau]^{1/2}} = \frac{1}{3r^2a^2}\left[-2r^3\right] = -\frac{2r}{3a^2} \qquad (1)$$

(ii) $r > a$

$$\int_{+1}^{-1} \frac{u\,du}{[r^2+a^2-2rau]^{1/2}} = \frac{1}{3r^2a^2}\left[-(r^2+a^2-ra)(r+a) + (r^2+a^2+ra)(r-a)\right]$$

$$= \frac{1}{3r^2a^2}\left[\underbrace{-r^3 - a^3 + r^3}_{0} + \overbrace{ra^2 + \underbrace{r^2a - r^2a}_{0} - a^3 - ra^2}^{0}\right]$$

$$= \frac{1}{3r^2a^2}\left[-2a^3\right] = \frac{-2a}{3r^2} \qquad (2)$$

Substituting the results of equations (1) & (2) into our expression for $\vec{A}(\vec{r})$, we get

$$A(r) = \frac{2\pi\sigma\omega a^3}{c}\sin\gamma \begin{cases} -\dfrac{2r}{3a^2} & r < a \\ -\dfrac{2a}{3r^2} & r > a \end{cases}$$

At this point, we can note that $\vec{\omega}\times\vec{r} = -\omega r \sin\gamma$ (see figure on first page of the problem). By making this substitution in $\vec{A}(\vec{r})$, then reorienting our coordinates such that $\vec{\omega}$ is aligned with the z-axis, we get that $\gamma \to \theta$ in our expression for A above (see Griffiths' example 5.11). Thus, we have the answer in what Griffiths calls natural coordinates.

$$\vec{A}(r,\theta) = \frac{2\pi\sigma\omega a^3}{c}\sin\theta \begin{cases} \dfrac{2r}{3a^2}\hat{\phi} & r \le a \\ \dfrac{2a}{3r^2}\hat{\phi} & r \ge a \end{cases}$$

$$\therefore \boxed{\vec{A}(r,\theta) = \frac{4\pi\sigma\omega}{3c}\sin\theta \begin{cases} ra\,\hat{\phi} & r \le a \\ \dfrac{a^4}{r^2}\hat{\phi} & r \ge a \end{cases}}$$

Now, we want to find the magnetic-flux density which as we saw in problem 5.6, is just the magnetic field.

In general, we have

$$\vec{B} = \nabla \times \vec{A}_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi)\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\,\hat{\theta}$$

So, for $r<a$

$$\vec{B} = \frac{4\pi\sigma\omega a}{3c}\left[\frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}(r^2)\,\hat{\theta}\right]$$

$$= \frac{4\pi\sigma\omega a}{3c}\left[\frac{2\sin\theta\cos\theta}{\sin\theta}\,\hat{r} - \frac{2r\sin\theta}{r}\,\hat{\theta}\right]$$

$$= \frac{8\pi\sigma\omega a}{3c}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

note: $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

Thus, $$\boxed{\vec{B} = \frac{8\pi\sigma\omega a}{3c}\hat{z}} \qquad r<a$$

For $r>a$

$$\vec{B} = \frac{4\pi\sigma\omega a^4}{3c}\left[\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}\right]$$

$$= \frac{4\pi\sigma\omega a^4}{3c}\left[\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\,\hat{r} + \frac{\sin\theta}{r^3}\,\hat{\theta}\right]$$

$$\Rightarrow \boxed{\vec{B} = \frac{4\pi\sigma\omega a^4}{3cr^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]} \qquad r>a$$

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Jackson 5.13

A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the spehre.

Write the current density in spherical coordinates:

$$\vec{J}(\vec{r}\,') = J_\phi \hat{\phi}\,' = [\sigma\,\omega\,r'\,\sin(\theta')\,\delta(r'-a)]\,\hat{\phi}\,'$$

Choosing the Coulomb gauge (thus can set the divergence of some scalar function to zero, see Jackson discussion p.181,) we write the vector potential using Jackson eq. 5.32. But from the azimuthal symmetry present in this case, we can evaluate the potential at $\phi = 0$ and the result will still be generally applicalbe. If we then proceed with the calculations using $\phi = 0$, we'd see that the only non-zero contribution to the potential is from the "y" component of the current density. Thus we have (in SI units):

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{J_\phi \cos(\phi')\hat{y}}{|\vec{r}-\vec{r}\,'|}\,(r')^2 dr'\,d\Omega'$$

We can immediately perform the r' integral using the delta function using the explicit expression for the current density. The result is:

$$\vec{A} = \hat{y}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\int \frac{\sin(\theta')\,\cos(\phi')\,d\Omega'}{|\vec{r}-a\hat{r}\,'|}$$

Now recall our discussion in the previous problem that the unit vector $\hat{y}$ for an azimuthally symmetric problem evaluated at $\phi = 0$ is generally equivalent ot $\hat{\phi}$. So we have:

$$\vec{A} = \hat{\phi}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\int \frac{\sin(\theta')\,\cos(\phi')\,d\Omega'}{|\vec{r}-a\hat{r}\,'|}$$

Now expand the inverse relative distance using spherical harmonics (Jackson eq. 3.70.) The vector potential expression becomes:

$$\vec{A} = \hat{\phi}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\int\left[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)\,Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi=0)\right]\sin(\theta')\cos(\phi')\,d\Omega'$$

where $r_<(r_>)$ is the smaller (larger) between r (field point) and a (source poiint). Now notice that

$$\sin(\theta')\cos(\phi') = \frac{1}{2}\sin(\theta')\left[e^{i\phi'} + e^{-i\phi'}\right] = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\left[-Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')\right]$$

where we've made use of Jackson eq. 3.54 and the definition for the spherical harmonics. With the above, we can perform the θ' and ϕ' integrals easily using the orthogonality property of the spherical harmonics. After a little manipulation, and using the fact that

$$Y_{11}(\theta,0) = -\sqrt{\frac{3}{8\pi}}\sin\theta = -Y_{1,-1}(\theta,0)$$

we arrive at the following result:

$$\vec{A} = \hat{\phi}\,\frac{\mu_0\sigma\omega a^3}{3}\left(\frac{r_<}{r_>^2}\right)\sin\theta$$

Hence the vector potentials both inside and outside the sphere are given by the following:

$$\vec{A}_{in} = \hat{\phi}\,\frac{\mu_0\sigma\omega a}{3}\,r\,\sin\theta$$

$$\vec{A}_{out} = \hat{\phi}\,\frac{\mu_0\sigma\omega a^4}{3}\,\frac{\sin\theta}{r^2}$$

Converting the above into gaussian units, we find:

$$\vec{A}_{in} = \hat{\phi}\,\frac{4\pi\sigma\omega a}{3c}\,r\,\sin\theta$$

$$\vec{A}_{out} = \hat{\phi}\,\frac{4\pi\sigma\omega a^4}{3c}\,\frac{\sin\theta}{r^2}$$

The magnetic flux density then is just $\vec{B} = \vec{\nabla}\times\vec{A}$. Straightforward calculations reveal the following:

$$\vec{B}_{in} = \frac{8\pi}{3c}\sigma\omega a\hat{z}$$

let $\sigma a\omega \to M$, $\boxed{\vec{B}_{in} = \frac{8\pi}{3c}M\hat{z}}$ (, recall $\mu_0 \to \frac{4\pi}{c}$)

b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_0 \hat{z}$. Using the above result, Find the magnetic field inside the sphere.

From Griffiths (eq. 6.29), we have (use MKS units ¨)

$$\vec{M} = \chi_m \vec{H} \quad , \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\Rightarrow \vec{M} = \frac{\chi_m}{\mu_0}\vec{B} - \chi_m \vec{M} \quad , \quad \vec{B} = \left(B_0 + \frac{2\mu_0}{3} M\right)\hat{z}$$

$$\Rightarrow \vec{M}(1+\chi_m) = \frac{\chi_m}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M}\chi_m$$

$$\therefore \quad \vec{M} = \frac{\chi_m \vec{B}_0}{\mu_0\left(1 + \frac{1}{3}\chi_m\right)}$$

Thus, the field inside the sphere is

$$\vec{B} = \vec{B}_0 + \frac{2\mu_0}{3}\left(\frac{\chi_m \vec{B}_0}{\mu_0(1+\frac{1}{3}\chi_m)}\right) = \vec{B}_0\left[1 + \frac{2\chi_m}{3+\chi_m}\right]$$

$$= \vec{B}_0\left[\frac{3+3\chi_m}{3+\chi_m}\right]$$

$$\therefore \quad \boxed{\vec{B} = \vec{B}_0\left[\frac{1+\chi_m}{1+(1/3)\chi_m}\right]}$$