Spring 2003 #1 (p 1 of 2)

In one-dimension, a particle is subject to a harmonic oscillator potential with a time dependent origin,

$$V(x) = \frac{1}{2} m\omega^2 \left[x - \epsilon(t)\right]^2$$

where

$$\epsilon(t) = \epsilon\, e^{-t^2/\tau^2} \qquad \epsilon \ll 1$$

Suppose the particle is in the ground state at $t = -\infty$. What states can the particle be in at $t = +\infty$, and what are the probabilities for each? Work to lowest order in ϵ.

So,

$$V(x) \cong \underbrace{\frac{1}{2} m\omega^2 x^2}_{= V(x)} \underbrace{- m\omega^2 x\, \epsilon(t)}_{= V'(x)} \qquad \leftarrow \text{to lowest order in } \epsilon$$

From Zettili eq. 10.41, we have that the transition probability is given by ($\hbar = 1$)

$$P_{if}(t) = \left| -i \int_0^t \langle \psi_f | V'(t') | \psi_i \rangle e^{i\omega_{fi} t'} dt' \right|^2$$

For our case, $t \to \infty$ and $|\psi_f\rangle = |n\rangle$, $|\psi_i\rangle = |0\rangle$, $\omega_{fi} = E_n - E_0$

$$= \omega(n + \tfrac{1}{2}) - \tfrac{\omega}{2}$$

$$= \omega n$$

So, we have

$$P_{n0}(t) = \left| \int_{-\infty}^{\infty} \langle n | (-m\omega^2 x\, \epsilon(t')) | 0 \rangle e^{i\omega n t'} dt' \right|^2 , \quad \epsilon(t) = \epsilon\, e^{-t^2/\tau^2}, \quad x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger)$$

$$= \frac{m^2 \omega^4}{2m\omega} \epsilon^2 \left| \int_{-\infty}^{\infty} \langle n | (a + a^\dagger) | 0 \rangle e^{-t'^2/\tau^2} e^{i\omega n t'} dt' \right|^2$$

where $\langle n | (a + a^\dagger) | 0 \rangle = \langle n | \cancel{a | 0 \rangle}^{\,0} + \langle n | a^\dagger | 0 \rangle = \sqrt{0+1}\, \delta_{n,1} = \delta_{n,1}$

Spring 2003 #1 (p 2 of 2)

So, we have $P_{n0}(t) = 0 \quad \forall n \neq 1$ and

$$P_{10}(t) = \left| \int_{-\infty}^{\infty} e^{-\left(\frac{1}{\tau^2} t'^2 - i\omega t'\right)} dt' \right|^2 \frac{m\omega^3}{2} \epsilon^2$$

note: $\int_{-\infty}^{\infty} e^{-(ax^2 + bx + c)} dx = \sqrt{\frac{\pi}{a}} \, e^{\frac{b^2 - 4ac}{4a}}$

so,

$$P_{10}(t) = \frac{m\omega^3}{2} \epsilon^2 \left| \sqrt{\frac{\pi}{1/\tau^2}} \, e^{(-\omega^2 - 0)\left(\frac{1}{4(\frac{1}{\tau^2})}\right)} \right|^2$$

$$= \frac{m\omega^3}{2} \epsilon^2 \tau^2 \pi \left| e^{-\frac{\omega^2 \tau^2}{4}} \right|^2$$

$$\therefore \boxed{P_{10}(t) = \frac{m\omega^3 \tau^2 \pi \epsilon^2}{2} e^{-\frac{\omega^2 \tau^2}{2}}}$$

all other $P_{n0}(t) = 0 \quad \forall n \neq 1$