A coherent state of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalue basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$.

(see Abus # 3.22 part (b))

we can rewrite the coherent state as follows: (Zettili eq 2.163)

$$|\alpha\rangle = I\,|\alpha\rangle = \left(\sum_{n=1}^{\infty} |n\rangle\langle n|\right)|\alpha\rangle$$

$$= \sum_{n=1}^{\infty} |n\rangle\,\langle n|\alpha\rangle = \sum_{n=1}^{\infty} a_n\,|n\rangle$$

where the coefficient a_n represents the projection of $|\alpha\rangle$ onto $|n\rangle$. a_n can be written in terms of a_0 by (see Zettili eq 4.137)

$$a_n = \frac{\alpha}{\sqrt{n}}\,c_{n-1} = \frac{\alpha^n}{\sqrt{n!}}\,c_0$$

to determine c_0, we need to use the normalization condition

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2\left(\sum_{n=1}^{\infty}\frac{|\alpha|^n}{\sqrt{n!}}\right)^2$$

note: $e^x = \sum_n \frac{x^n}{n!} \Rightarrow e^{x^2} = \sum_n \frac{x^{2n}}{n!}$

so,

$$|c_0|^2 = e^{-\alpha^2} \Rightarrow c_0 = e^{-\frac{\alpha^2}{2}}$$

Thus,

$$\boxed{|\alpha\rangle = e^{-\frac{\alpha^2}{2}}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle}$$