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Two spin-half particles are in a state with total spin zero. Let $\hat{n}_a$ and $\hat{n}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the product of the spin of the first particle along $\hat{n}_a$ and the spin of the second along $\hat{n}_b$. That is, if $\vec{S}_a$ and $\vec{S}_b$ are the two spin operators, calculate

$$\langle\psi|\vec{S}_a\cdot\hat{n}_a\ \vec{S}_b\cdot\hat{n}_b|\psi\rangle$$

(See Abers 4.11)

From the definition of the dot product

$$\vec{A}\cdot\vec{B}=\sum_i A_i B_i$$

So, we can write

$$\gamma\equiv\langle\psi|\vec{S}_a\cdot\hat{n}_a\vec{S}_b\cdot\hat{n}_b|\psi\rangle=\langle\psi|\sum_{ij}(S_a)_i(n_a)_i(S_b)_j(n_b)_j|\psi\rangle$$

since $(n_a)_i$ and $(n_b)_j$ are just numbers, we get

$$\gamma=\sum_{ij}(n_a)_i(n_b)_j\langle\psi|(S_a)_i(S_b)_j|\psi\rangle \qquad (1)$$

Now, $(S_a)_i(S_b)_j$ has the form of a 2nd rank tensor

$$T_{ij}=(S_a)_i(S_b)_j$$

So, from Abers eq 5.40, we can write the tensor as

$$T_{ij}=\underbrace{\left[\frac{1}{3}\delta_{ij}\sum_K T_{KK}\right]}_{\text{trace, spin zero}}+\underbrace{\left[\frac{1}{2}(T_{ij}-T_{ji})\right]}_{\text{anti-symmetric part}}+\underbrace{\left[\frac{1}{2}(T_{ij}+T_{ji})-\frac{1}{3}\delta_{ij}\sum_K T_{KK}\right]}_{\text{traceless, symmetric part}}$$

Since the spin of our system is equal to zero, only the 1st term on the RHS survives. So,

$$T_{ij}=\frac{1}{3}\delta_{ij}\sum_K T_{KK}=\frac{1}{3}\sum_K(S_a)_K(S_b)_K=\frac{1}{3}\vec{S}_a\cdot\vec{S}_b$$

note:

$$\underbrace{[(S_a+S_b)^2]_K}_{\text{total spin is zero}} = (S_a^2)_K + (S_b^2)_K + 2(\vec{S}_a \cdot \vec{S}_b)_K$$

$$\Rightarrow \quad (\vec{S}_a \cdot \vec{S}_b)_K = -\frac{1}{2}\left[(S_a^2)_K + (S_b^2)_K\right]$$

Now, take advantage of us having spin half particles. That is,

$$S_i = \frac{\sigma_i}{2}$$

$$\Rightarrow \quad (\vec{S}_a \cdot \vec{S}_b)_K = -\frac{1}{8}\left[(\sigma_a^2)_K + (\sigma_b^2)_K\right]$$

we know that a property of the Pauli matrices is that $\sigma_i^2 = 1$ (Abers 4.77). So,

$$(\vec{S}_a \cdot \vec{S}_b)_K = -\frac{1}{8}[1+1] = -\frac{1}{4}$$

Summing over the three coordinates, we get

$$T_{ij} = \frac{1}{3}\sum_K (\vec{S}_a \cdot S_b)_K = \frac{1}{3}\left(-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}\right) = -\frac{1}{4}$$

substituting this result into eq (1) yields

$$\sum_{i,j}(n_a)_i(n_b)_j \langle\psi|(S_a)_i(S_b)_j|\psi\rangle = \sum_i (n_a)_i(n_b)_i\left(-\frac{1}{4}\right)\underbrace{\langle\psi|\psi\rangle}_{\text{normalized orthogonal vectors}}$$

$$= -\frac{1}{4}\,\hat{n}_a \cdot \hat{n}_b = -\frac{1}{4}|\hat{n}_a||\hat{n}_b|\cos(\theta_{ab})$$

where θ_{ab} is the angle between $\hat{n}_a$ and $\hat{n}_b$. Thus,

$$\boxed{\langle\psi|\vec{S}_a \cdot \hat{n}_a\, \vec{S}_b \cdot \hat{n}_b|\psi\rangle = -\frac{1}{4}\cos(\theta_{ab})}$$

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Abers solution (#4.11)

1. *Quantum Mechanics*

Two spin-half particles are in a state with total spin zero. Let $\hat{n}_a$ and $\hat{n}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{n}_a$ and the spin of the second along $\hat{n}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\,\mathbf{s}_a\cdot\hat{n}_a\mathbf{s}_b\cdot\hat{n}_b\,|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

This is an example of the selection rules from section 5.2. Since the state ψ is symmetric,

$$\begin{aligned}\langle\psi|\left(\mathbf{s}_a\cdot\hat{\boldsymbol{n}}_a\right)\left(\mathbf{s}_b\cdot\hat{\boldsymbol{n}}_b\right)|\psi\rangle &= \sum_{ij}\left(\hat{\boldsymbol{n}}_a\right)_i\left(\hat{\boldsymbol{n}}_b\right)_j\left\langle\psi\middle|\left(s_a\right)_i\left(s_b\right)_j\middle|\psi\right\rangle \\ &= \sum_{ij}\left(\hat{\boldsymbol{n}}_a\right)_i\left(\hat{\boldsymbol{n}}_b\right)_j\left\langle\psi\middle|\frac{1}{3}\delta_{ij}\sum_k\left(s_a\right)_k\left(s_b\right)_k\middle|\psi\right\rangle+\cdots\end{aligned} \tag{S10.28}$$

The remaining terms are matrix elements of (linear combinations of) components of spherical tensors of ranks 1 and 2 between spin-zero states, so vanish:

$$\langle\psi|\left(\mathbf{s}_a\cdot\hat{\boldsymbol{n}}_a\right)\left(\mathbf{s}_b\cdot\hat{\boldsymbol{n}}_b\right)|\psi\rangle = \frac{1}{3}\hat{\boldsymbol{n}}_a\cdot\hat{\boldsymbol{n}}_b\langle\psi|\,\mathbf{s}_a\cdot\mathbf{s}_b\,|\psi\rangle \tag{S10.29}$$

But since $\mathbf{s}^2=0$ between these states, where $\mathbf{s}=\mathbf{s}_a+\mathbf{s}_b$,

$$\mathbf{s}_a\cdot\mathbf{s}_b=\frac{1}{2}\left(\mathbf{s}^2-\mathbf{s}_a^2-\mathbf{s}_b^2\right)=-\frac{3}{4} \tag{S10.30}$$

and

$$\langle\psi|\left(\mathbf{s}_a\cdot\hat{\boldsymbol{n}}_a\right)\left(\mathbf{s}_b\cdot\hat{\boldsymbol{n}}_b\right)|\psi\rangle=-\frac{1}{3}\cos\theta\frac{3}{4}=-\frac{1}{4}\cos\theta \tag{S10.31}$$

Note: It is of course possible to get the answer without using the theorem: Since the matrix element depends only on the angle between these two directions, let $\hat{\boldsymbol{n}}_a=\hat{\boldsymbol{n}}_z$. Then with $\hat{\boldsymbol{n}}_b=\cos\theta\hat{\boldsymbol{n}}_z+\sin\theta\hat{\boldsymbol{n}}_x$, the correlation is

$$\begin{aligned}E(\theta)=\langle\psi|\left(\mathbf{s}_a\cdot\hat{\boldsymbol{n}}_z\right)\left(\mathbf{s}_b\cdot\hat{\boldsymbol{n}}_c\right)|\psi\rangle &= \frac{1}{4}\langle\psi|\left(\boldsymbol{\sigma}_a\cdot\hat{\boldsymbol{n}}_z\right)\left(\boldsymbol{\sigma}_b\cdot\hat{\boldsymbol{n}}_c\right)|\psi\rangle \\ &= \langle\psi|\left(\boldsymbol{\sigma}_a\right)_z\left[\left(\boldsymbol{\sigma}_b\right)_z\cos\theta+\left(\boldsymbol{\sigma}_b\right)_x\sin\theta\right]|\psi\rangle\end{aligned} \tag{S10.32}$$

[2] see Equation (A.27) in the appendix.

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(Aber's solution)

[redacted]

Now

$$(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)|\psi\rangle = (\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)\frac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z \frac{|++\rangle - |--\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}} \tag{S10.33}$$

so that

$$\langle\psi|(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)|\psi\rangle = 0 \tag{S10.34}$$

Similarly

$$(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)|\psi\rangle = (\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)\frac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \frac{-|+-\rangle + |-+\rangle}{\sqrt{2}} = -|\psi\rangle \tag{S10.35}$$

so that

$$\langle\psi|(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)|\psi\rangle = -1 \tag{S10.36}$$

So again

$$E(\theta) = -\frac{1}{4}\cos\theta \tag{S10.37}$$