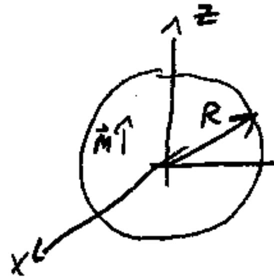
Fall 2004 #8 (p 1 of 3)

Consider a sphere of radius a with uniform magnetization $\vec{M}$, pointing in the z-direction. What are the magnetic induction $\vec{B}$ and magnetic field $\vec{H}$ inside the sphere?

(see spring 2003 #12 and Jackson problem 5.13)

First, recognize that this is an identical problem to the field of a spinning spherical shell with $\sigma R\omega \to M$ (see Griffiths' examples 6.1 and 5.11). That is, we have



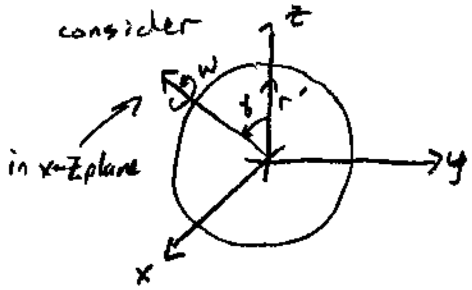
$\vec{j}_m = \nabla\times\vec{M} = 0$
$|\vec{K}| = M\sin\theta$ } mks units

if rotating spherical shell of uniform charge, then
$K = \sigma\omega R\sin\theta$

So let's solve the rotating spherical shell of uniform charge. First we must find the vector potential since

$$\vec{A} = \frac{1}{c}\int \frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,da' \qquad (1)$$

consider



$$\text{so,}\quad \underset{\substack{\uparrow\\ \vec{v}=\vec{\omega}\times\vec{r}\,'}}{\vec{K} = \sigma\vec{v}} = \underset{\substack{\uparrow\\ \text{axis of rotation}\\ x\text{-}z\text{ plane}}}{\sigma(\vec{\omega}\times\vec{r}\,')} = \sigma\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w\sin\gamma & 0 & w\cos\gamma \\ a\sin\theta'\sin\phi' & a\sin\theta'\sin\phi' & a\cos\theta \end{vmatrix}$$

$$\Rightarrow \vec{K} = \sigma\Big[\hat{x}(-wa\cos\gamma\sin\theta'\sin\phi') + \hat{y}(wa\cos\gamma\sin\theta'\sin\phi' - wa\sin\gamma\cos\theta) + \hat{z}(aw\sin\gamma\sin\theta'\sin\phi')\Big]$$

Now note; $|\vec{r}-\vec{r}\,'| = \left[r^2+(r')^2-2rr'\cos\theta'\right]^{1/2}\Big|_{r'=a} = \left[r^2+a^2-2ra\cos\theta'\right]^{1/2}$

and $da' = a^2\sin\theta'\,d\theta'\,d\phi'$

Substituting these results into eq (1) yields

$$\vec{A}(\vec{r}) = \frac{\sigma w a^3}{c}\int_0^{2\pi}d\phi'\int_0^{\pi}\sin\theta'\,d\theta'\left[\frac{-\sin\theta'\sin\phi'\cos\gamma\,\hat{x} + (\sin\theta'\cos\phi'\cos\gamma - \sin\gamma\cos\theta')\hat{y} + \sin\gamma\sin\theta'\sin\phi'}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}\right.$$

$$\text{since}\quad \int_0^{2\pi}\sin\phi'\,d\phi' = -\left[\cos\phi'\right]_0^{2\pi} = 0$$

$$\text{and}\quad \int_0^{2\pi}d\phi'\cos\phi' = \left[\sin\phi'\right]_0^{2\pi} = 0$$

the integration over ϕ' in the x and z direction vanish as well as the first term in the y direction.

So, this messy integral reduces to

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \left[\frac{-\sin\gamma\cos\theta'}{[r^2+a^2-2ra\cos\theta']^{1/2}} \right]$$

$$= \frac{-2\pi\sin\gamma\,\sigma\omega a^3}{c}\int_0^{\pi} d\theta' \frac{\sin\theta'\cos\theta'}{[r^2+a^2-2ra\cos\theta']^{1/2}}$$

let $u = \cos\theta' \Rightarrow du = -\sin\theta' d\theta'$

So, we have

$$\vec{A}(\vec{r}) = \frac{-2\pi\sigma\omega a^3}{c}\sin\gamma \int_{-1}^{1} \frac{u\,du}{[r^2+a^2-2rau]^{1/2}}$$

note: $\displaystyle\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$

So,

$$\vec{A}(\vec{r}) = \frac{-2\pi\sigma\omega a^3}{c}\sin\gamma\left(\frac{-2r}{3a^2}\right)\hat{\phi} \qquad r < a$$

(→ see p 4 and 5 of the Spring 2003 #12 for details of this calculation)

recall that $\vec{\omega}\times\vec{r} = -\omega r \sin\gamma$ from figure on p 1. So,

$$\vec{A}(\vec{r}) = \frac{2\pi\sigma\omega a^3}{c}\sin\theta\left(\frac{2r}{3a^2}\right) = \frac{4\pi\sigma\omega r a}{3c}\sin\theta\,\hat{\phi}$$

where we re-oriented our coordinate system such that $\vec{\omega}$ is aligned with the z-axis,

Now, we are ready to find $\vec{B}$ where $\vec{B} = \nabla\times\vec{A} = \nabla\times(A_\phi\hat{\phi})$

$$\Rightarrow \vec{B} = \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix} \frac{1}{r^2\sin\theta} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\theta}$$

$$= \frac{4\pi\sigma\omega a}{3c}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}r^2\,\hat{\theta}\right] = \frac{4\pi\sigma\omega a}{3c}\left[\frac{2\sin\theta\cos\theta}{\sin\theta}\hat{r} - \frac{2r}{r}\sin\right.$$

$$= \frac{8\pi\sigma\omega a}{3c}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] = \frac{8\pi\sigma\omega a}{3c}\hat{z}$$

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Thus,

$$\boxed{\vec{B} = \frac{8\pi\sigma\omega a}{3c}\hat{z}} \qquad r < a$$

Now, find $\vec{H}$ inside.

$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad , \quad \vec{M} = M_0\hat{z}$$

Thus,

$$\boxed{\vec{H} = \hat{z}\left[\frac{8\pi\sigma\omega a}{3c} - 4\pi M_0\right]}$$