QM S'04 #2

H-atom is in the ground state ($n=1, \ell=m=0$) at $t<0$

A time dependent E-field is applied:

$$\vec{E} = \vec{E}_0 e^{-\gamma t} \quad \gamma > 0 \; ; \; \vec{E}_0 = E_0 \hat{z}$$

What is the probability that for $t \to \infty$ the atom is in each of the four $n=2$ states?

Start: $n=1, \ell=0=m$ $\quad |1,0,0\rangle$

Finish $n=2$; $\ell=0$ $|2,0,0\rangle$; $\ell=1$ $\quad |2,1,1\rangle, |2,1,0\rangle, |2,1,-1\rangle$

$\quad m=0 \qquad\qquad m = 1,0,-1$

$$c_{1\to2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle H'\rangle e^{i\omega_0 t}\,dt \; ; \; \omega_0 = \frac{E_2 - E_1}{\hbar} \; ; \; E_1 = -13.6\,eV, \quad E_2 = \frac{-13.6\,eV}{4}$$

$$= -\frac{13.6\,eV}{\hbar}\left(\frac{1}{4}-1\right) = \frac{13.6\,eV}{\hbar}\cdot\frac{3}{4}$$

$\langle H'\rangle = \langle 100|H'|200\rangle ; \langle 100|H'|211\rangle ; \langle 100|H'|210\rangle ; \langle 100|H'|21{-1}\rangle$

$$\bullet \; \langle 100|H'|200\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{2e^{-r/a}}{\sqrt{4\pi}\,a^{3/2}}\right) E_0 \overset{r\cos\theta}{\downarrow}{z}\, e^{-\gamma t}\left(\frac{1}{\sqrt{2}a^{3/2}\sqrt{4\pi}}\left(1-\frac{r}{2a}\right)e^{-r/2a}\right) r^2\sin\theta\,d\theta\,d\phi\,dr$$

$$= \frac{2}{4\pi a^3\sqrt{2}} E_0 e^{-\gamma t} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi}\cos\theta\sin\theta\,d\theta}_{} \int_0^\infty r^3\left(1-\frac{r}{2a}\right)e^{-\frac{3r}{2a}}dr \; \underline{= 0}$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$\Rightarrow \int_0^0 u\,du = 0$$

So $\underline{\langle 100|H'|200\rangle = 0}$

$\cdot\ \langle 100|H'|210\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\cancel{2}e^{-r/a}}{a^{3/2}\sqrt{4\pi}}\left(E_0 r\cos\theta\, e^{-\gamma t}\right)\left(\frac{1}{\cancel{2}\sqrt{6}}\frac{r}{a\,a^{3/2}}e^{-r/2a}\sqrt{\frac{3}{4\pi}}\cos\theta\right) r^2 dr\,\sin\theta\, d\theta\, d\phi$

$$= \frac{E_0 e^{-\gamma t}}{4\pi a^3}\frac{1}{\sqrt{2}}\frac{1}{a}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta}\underbrace{\int_0^{\infty} r^4 e^{-\frac{3r}{2a}}dr} = \frac{E_0 e^{-\gamma t}}{\cancel{4\pi}a^4}\frac{1}{\sqrt{2}}\cdot\cancel{2\pi}\cdot\frac{2}{3}\cdot\left(\frac{2a}{3}\right)^5\cdot 4!$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-\int_1^{-1} u^2 du = \int_{-1}^{1} u^2 du = \frac{1}{3}u^3\Big|_{-1}^{1} = \frac{1}{3}(1-(-1)) = \frac{2}{3}$

$\int_0^{\infty} x^n e^{-x/a'}dx = a'^{n+1} n! \quad n = 4;\ a' = \frac{2a}{3}$

$= \left(\frac{2a}{3}\right)^5 4!$

So $\langle 100|H'|210\rangle = \frac{E_0 e^{-\gamma t}}{3\sqrt{2}a^4}\left(\frac{2a}{3}\right)^5 4! = \boxed{\frac{E_0 e^{-\gamma t}a}{\sqrt{2}}8\left(\frac{2}{3}\right)^5}$

$\cdot\langle 100|H'|211\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\cancel{2}e^{-r/a}}{a^{3/2}\sqrt{4\pi}}\left(E_0 r\cos\theta\, e^{-\gamma t}\right)\left(\frac{1}{\cancel{2}\sqrt{6}}\frac{r}{a\,a^{3/2}}e^{-r/2a}\cdot -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right) r^2 dr \sin\theta\, d\theta\, d\phi$

$$= \frac{-E_0 e^{-\gamma t}}{a^4 4\pi}\frac{1}{\sqrt{4}}\underbrace{\int_0^{2\pi} e^{i\phi}d\phi}\int_0^{\pi}\cos\theta\sin^2\theta\, d\theta\int_0^{\infty} r^4 e^{-\frac{3r}{2a}}dr = 0$$

$\frac{1}{i}e^{i\phi}\Big|_0^{2\pi} = \frac{1}{i}[1-1] = 0$

So $\langle 100|H'|211\rangle = 0$ and similarly for $\langle 100|H'|21-1\rangle$.

So $c_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^{\infty} 8\frac{E_0 e^{-\gamma t}a}{\sqrt{2}}\left(\frac{2}{3}\right)^5 e^{i\omega_0 t}dt = \frac{-i E_0 a 8}{\hbar\sqrt{2}}\left(\frac{2}{3}\right)^5\underbrace{\int_0^{\infty} e^{-(\gamma - i\omega_0)t}dt}$

$\frac{1}{-(\gamma - i\omega_0)}e^{-(\gamma-i\omega_0)t}\Big|_0^{\infty} = \frac{+1}{(\gamma - i\omega_0)}$

$c_{1\to 2}(t) = \frac{-i E_0 a 8}{\hbar\sqrt{2}}\left(\frac{2}{3}\right)^5\frac{1}{(\gamma - i\omega_0)}$

So the probability is: $|c_{1\to 2}(t)|^2 = \frac{E_0^2 a^2}{2\hbar^2}\left(\frac{2}{3}\right)^{10}\frac{8^2}{(\gamma - i\omega_0)^2} = \boxed{\frac{E_0^2 a^2}{2\hbar^2}\left(\frac{2}{3}\right)^{10}\frac{64}{\gamma^2+\omega_0^2}}$

$\uparrow$

$\frac{8}{\gamma^2+\omega_0^2}$