QM S'04 #3

$$\psi(x) = N e^{-Kx^2/2} \quad ; \quad K > 0$$

a) $N = ?$

$$1 = \int_{-\infty}^{\infty} N e^{-Kx^2/2} N e^{-Kx^2/2} dx = N^2 \int_{-\infty}^{\infty} e^{-Kx^2} dx = N^2 \sqrt{\frac{\pi}{K}}$$

$$2\int_0^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \quad N = \left(\frac{K}{\pi}\right)^{1/4}$$

b) $$\langle x^2 \rangle = N^2 \int_{-\infty}^{\infty} x^2 e^{-Kx^2} dx = \frac{N^2 \sqrt{\pi}}{2K^{3/2}} = \frac{K^{1/2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2K^{3/2}} = \frac{K^{-1}}{2} = \frac{1}{2K}$$

$$2\int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{a^{(m+1)/2}} \quad ; \quad m = 2; \quad a = K$$

$$= \frac{\sqrt{\pi}}{2K^{3/2}}$$

$$\langle x^2 \rangle = \frac{1}{2K}$$

c) $$\langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \psi(x) dx = \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} e^{-Kx^2/2} dx$$

$$= \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\left(\frac{K}{2}x^2 + \frac{ip}{\hbar}x\right)} dx$$

Let $y = \sqrt{a}\, x + \frac{b}{2\sqrt{a}} \Rightarrow y^2 = ax^2 + bx + \frac{b^2}{4a}$; $dy = \sqrt{a}\, dx$

$dx = \frac{dy}{\sqrt{a}}$

$a = \frac{K}{2}$; $b = \frac{ip}{\hbar}$ This is completing the square

$$\langle p|\psi\rangle = \frac{N}{\sqrt{2\pi\hbar}}\frac{e^{b^2/4a}}{\sqrt{a}}\underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{\sqrt{\pi}} = \frac{N}{\sqrt{2\pi a\hbar}}\sqrt{\pi}\,e^{b^2/4a}$$

$$= \frac{(k/\pi)^{1/4}}{\sqrt{2a\hbar}}\,e^{b^2/4a} \quad ; \quad a = \frac{K}{2} \; ; \; b = \frac{ip}{\hbar}$$

so $\dfrac{b^2}{4a} = \dfrac{(ip/\hbar)^2}{2\cancel{4}\frac{K}{\cancel{2}}} = \boxed{\dfrac{-p^2}{2\hbar^2 K}}$ and $\dfrac{(K/\pi)^{1/4}}{\sqrt{\hbar}\,K^{1/2}} = \dfrac{1}{\sqrt{\hbar}(\pi K)^{1/4}}$

$$\therefore \quad \langle p|\psi\rangle = \frac{e^{-\frac{p^2}{2\hbar^2 K}}}{\sqrt{\hbar}(\pi K)^{1/4}}$$

d) $$\langle p^2\rangle = \frac{1}{\hbar\sqrt{\pi K}}\int_{-\infty}^{\infty} p^2 e^{-p^2/\hbar^2 K}\,dp = \frac{1}{\hbar\sqrt{\pi K}}\frac{\sqrt{\pi}(\hbar^2 K)^{3/2}}{2} = \frac{\hbar^2 K}{2}$$

$$\uparrow$$

$$2\int_0^\infty x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{a^{(m+1)/2}} = \frac{\frac{\sqrt{\pi}}{2}}{\left(\frac{1}{\hbar^2 K}\right)^{3/2}} = \frac{\sqrt{\pi}}{2}(\hbar^2 K)^{3/2}$$

e) $H\psi = E\psi$ need to assume this:

$$p = -i\hbar\frac{d}{dx} \Rightarrow p^2 = -\hbar^2\frac{d^2}{dx^2}$$

$$\frac{p^2}{2m}\psi + V(x)\psi = E\psi \Rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi = E\psi$$

$$\frac{d^2}{dx^2}\psi = \frac{d}{dx}\frac{d}{dx}\psi = \frac{d}{dx}\left(\frac{d}{dx}Ne^{-Kx^2/2}\right) = N\frac{d}{dx}\left(-Kxe^{-Kx^2/2}\right) = -KN\left(-Kx^2e^{-Kx^2/2} + e^{-Kx^2/2}\right)$$

$$= -K(-Kx^2+1)Ne^{-Kx^2/2} = -K(-Kx^2+1)\psi$$

$$\Rightarrow \frac{-\hbar^2}{2m}\left[-K(-Kx^2+1)\right]\cancel{\psi} + V(x)\cancel{\psi} = E\cancel{\psi} \Rightarrow -\frac{\hbar^2K^2x^2}{2m} + \frac{\hbar^2 K}{2m} + V(x)\cancel{\psi} = E$$

so $$V(x) - \frac{\hbar^2K^2x^2}{2m} = E - \frac{\hbar^2}{2m}K \quad \text{for all } x \therefore$$

or $$V(x) - \frac{\hbar^2K^2}{2m}x^2 = \text{const.} \Rightarrow V(x) = \frac{\hbar^2K^2}{2m}x^2 + \text{const.}$$

same

$$E(x) - \frac{\hbar^2}{2m}K = \text{const.} \Rightarrow E(x) = \frac{\hbar^2}{2m}K + \text{const.}$$