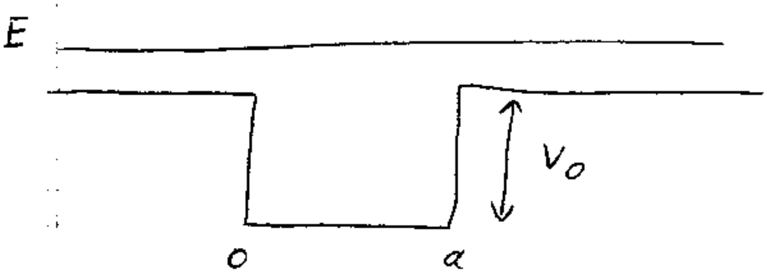
QM S'04 #5



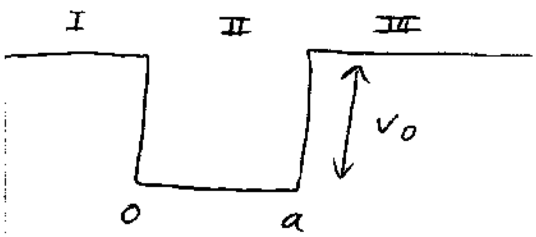
$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

resonance is when $T = 1$ which happens when

$$\frac{a}{\hbar}\sqrt{2m(E+V_0)} = n\pi \quad \text{or} \quad 2m(E+V_0) = \frac{n^2\pi^2\hbar^2}{a^2}$$

$$\boxed{E + V_0 = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

Or the derivation:



$E > 0$

I) $V(x) = 0 \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi \qquad k_1 \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$= -k_1^2\psi$$

$$\Rightarrow \psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad \text{for} \quad x < 0$$

by a similar argument for region III we end up with

$$\psi(x) = F e^{ik_1 x} + G e^{-ik_1 x} \qquad x > a$$

For region II $\quad V(x) = -V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\psi = E\psi \Rightarrow \frac{d^2}{dx^2}\psi = -\frac{2m(E+V_0)}{\hbar^2}\psi \; ; \; k_2 \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\frac{d^2}{dx^2}\psi = -k_2^2\psi \Rightarrow \quad \psi(x) = Ce^{ik_2 x} + De^{-ik_2 a} \qquad \text{for } 0<x<a$$

In summary

$$\psi(x) = \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x} & x<0 \\ Ce^{ik_2 x} + De^{-ik_2 x} & 0<x<a \\ Fe^{ik_1 x} + Ge^{-ik_1 x} & x>a \end{cases}$$

Now we need to match up the wave functions on the boundary conditions:

$x = 0$

$$A + B = C + D \qquad (1)$$

$$\left.\frac{d}{dx}\right|_{x=0} \quad ik_1 A - ik_1 B = ik_2 C - ik_2 D \qquad (2)$$

$$(2)/k_1 \qquad A - B = \frac{k_2}{k_1}(C - D) \qquad (2')$$

$$(1) + (2') \qquad 2A = \left(1 + \frac{k_2}{k_1}\right)C + \left(1 - \frac{k_2}{k_1}\right)D$$

$$A = \frac{1}{2}\left\{\left(1 + \frac{k_2}{k_1}\right)C + \left(1 - \frac{k_2}{k_1}\right)D\right\}$$

$$(1) - (2') \qquad 2B = \left(1 - \frac{k_2}{k_1}\right)C + \left(1 + \frac{k_2}{k_1}\right)D$$

$$B = \frac{1}{2}\left\{\left(1 - \frac{k_2}{k_1}\right)C + \left(1 + \frac{k_2}{k_1}\right)D\right\}$$

QM S'04 #5

So we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\frac{k_2}{k_1}) & (1-\frac{k_2}{k_1}) \\ (1-\frac{k_2}{k_1}) & (1+\frac{k_2}{k_1}) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

Now for the other boundary:

$x=a$

$$Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a} + Ge^{-ik_1a} \qquad (3)$$

$$\left.\frac{d}{dx}\right|_{x=a} \quad ik_2Ce^{ik_2a} - ik_2De^{-ik_2a} = ik_1Fe^{ik_1a} - ik_1Ge^{-ik_1a} \qquad (4)$$

$$(4)/k_2 \qquad Ce^{ik_2a} - De^{-ik_2a} = \frac{k_1}{k_2}\left(Fe^{ik_1a} - Ge^{-ik_1a}\right) \qquad (4')$$

$$(3)+(4) \qquad 2Ce^{ik_2a} = (1+\frac{k_1}{k_2})Fe^{ik_1a} + (1-\frac{k_1}{k_2})Ge^{-ik_1a}$$

$$C = \frac{1}{2}\left\{(1+\frac{k_1}{k_2})Fe^{i(k_1-k_2)a} + (1-\frac{k_1}{k_2})Ge^{-i(k_1+k_2)a}\right\}$$

$$(3)-(4) \qquad 2De^{-ik_2a} = (1-\frac{k_1}{k_2})Fe^{ik_1a} + (1+\frac{k_1}{k_2})Ge^{-ik_1a}$$

$$D = \frac{1}{2}\left\{(1-\frac{k_1}{k_2})Fe^{i(k_1+k_2)a} + (1+\frac{k_1}{k_2})Ge^{i(k_2-k_1)a}\right\}$$

So

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\frac{k_1}{k_2})e^{i(k_1-k_2)a} & (1-\frac{k_1}{k_2})e^{i(k_1+k_2)a} \\ (1-\frac{k_1}{k_2})e^{i(k_1+k_2)a} & (1+\frac{k_1}{k_2})e^{-i(k_2-k_1)a} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}$$

Combining the two matrices in order to get A, B in terms of F, G:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\frac{k_2}{k_1}) & (1-\frac{k_2}{k_1}) \\ (1-\frac{k_2}{k_1}) & (1+\frac{k_2}{k_1}) \end{pmatrix} \frac{1}{2}\begin{pmatrix} (1+\frac{k_1}{k_2})e^{i(k_1-k_2)a} & (1-\frac{k_1}{k_2})e^{-i(k_1+k_2)a} \\ (1-\frac{k_1}{k_2})e^{i(k_1+k_2)a} & (1+\frac{k_1}{k_2})e^{-i(k_2-k_1)a} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}$$

Now what we care for is the transmission coefficient which is :

$$T = \frac{|F|^2}{|A|^2} \quad \text{or} \quad T^{-1} = \frac{|A|^2}{|F|^2}$$

now the latter is more useful as the above matrix has A in terms of F. So we need to multiply the first row by the first column to get what we want:

$$A = \frac{1}{4}\left[(1+\frac{k_2}{k_1})(1+\frac{k_1}{k_2})e^{i(k_1-k_2)a} + (1-\frac{k_2}{k_1})(1-\frac{k_1}{k_2})e^{i(k_1+k_2)a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[(1+\frac{k_1}{k_2}+\frac{k_2}{k_1}+1)e^{-ik_2a} + (1-\frac{k_1}{k_2}-\frac{k_2}{k_1}+1)e^{ik_2a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[\underbrace{2e^{-ik_2a}} + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)e^{-ik_2a} + \underbrace{2e^{ik_2a}} - \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)e^{ik_2a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[4\cos(k_2a) \mp \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\underbrace{\left[e^{-ik_2a} - e^{-ik_2a}\right]}_{2i\sin(k_2a)}\right]F$$

$$\text{Now } |A|^2 = \frac{\overbrace{e^{ik_1}e^{-ik_1}}^{=1}}{16}\left[16\cos^2(k_2a) - \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\left[\cancel{2i\sin(k_2a)\cos(k_2a)} - \cancel{2i\sin(k_2a)\cos(k_2a)}\right] + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)^2 4\sin^2(k_2a)\right]|F|^2$$

QM S'04 #5

So $|A|^2 = \frac{1}{16}\left[16\cos^2(k_2 a) + 4\frac{(k_1^2+k_2)^2}{k_1^2 k_2^2}\sin^2(k_2 a)\right]|F|^2$

$$= \left[\cos^2(k_2 a) + \frac{(k_1^2+k_2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)\right]|F|^2$$

$$= \left[\frac{4k_1^2k_2^2\cos^2(k_2 a) + (k_1^2+k_2^2)^2\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[\frac{4k_1^2k_2^2 - 4k_1^2k_2^2\sin^2(k_2 a) + k_1^4\sin^2(k_2 a) + k_2^4\sin^2(k_2 a) + 2k_1^2k_2^2\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[1 + \frac{(k_1^4 + k_2^4 - 2k_1^2k_2^2)\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[1 + \frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)\right]|F|^2$$

So $T^{-1} = \frac{|A|^2}{|F|^2} = 1 + \frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)$

now $k_1 = \frac{\sqrt{2mE}}{\hbar}$; $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

so $\frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2} = \frac{\left(\frac{2mE - 2mE - 2mV_0}{\hbar^2}\right)^2}{4\left(\frac{4m^2E(E-V_0)}{\hbar^4}\right)} = \frac{4m^2V_0^2}{4(4m^2)(E(E-V_0))} = \frac{V_0^2}{4E(E-V_0)}$

$$\sin^2(k_2 a) = \sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)$$

hence

$$T^{-1} = 1 + \frac{V_0^2}{4E(E-V_0)}\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)$$

and for $T=1$ $\sin^2(\ldots)=0$ or $\frac{a}{\hbar}\sqrt{2m(E-V_0)} = n\pi$

$\Rightarrow E + V_0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$