Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\vec{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(See spring 2001 #13 and Erik's wonderful explanation of general BEC problems for massive/massless bosons in d-dimensions)

Since the energy is given by $E = |\vec{p}|c$, we assume that we are talking about massless particles. Let's further assume that they are spin 0, so, the degeneracy is one.

The procedure is to find the transition temperature at which a BEC forms. We can get an expression for the transition temperature from the expression for the total number of bosons.

$$N = \int_0^\infty \bar{n}(\epsilon)\, dN \quad , \quad dN = \underbrace{D(\epsilon)}_{\text{density of states}} d\epsilon \qquad (1)$$

The convention used to find the density of states is to take a very large cube (if 3-D) each of side L and force the wave functions representing the bosons to vanish at the walls. This leads to the condition for the quantized wave vector to be

$$k_i = \frac{n_i \pi}{L} \quad , \quad i = x, y, z$$

So, in 3D with $\epsilon = |\vec{p}|c$, we have

$$\epsilon = |\vec{p}|c = \hbar|\vec{k}|c = \frac{\hbar c \pi}{L} n_i$$

Solving for n_i yields

$$n_i = \frac{\epsilon L}{\hbar c \pi}$$

Then the density of states (in n space) is given by (in 3-D)

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = 4\pi n^2 \frac{dn}{d\epsilon}$$

So,

$$D(\epsilon) = 4\pi\left(\frac{\epsilon L}{\hbar c \pi}\right)^2 \frac{L}{\hbar c \pi} = 4\pi\left(\frac{L}{\hbar c \pi}\right)^3 \epsilon^2$$

Let $\hbar = c = 1$ $\Rightarrow$ $\boxed{D(\epsilon) = \frac{4L^3}{\pi^2}\epsilon^2}$

Substituting this result into eq (1) for the total N yields

$$N_{3D} = \frac{4L^3}{8\pi^2}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon$$

where we used $\bar{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$ for bosons and a factor of "$\frac{1}{8}$" because we only care about the positive values of the sphere in "n-space" which is $1/8$ of the total sphere. So, we have

$$N_{3D} = \frac{L^3}{2\pi^2}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon = \frac{V}{\pi^2\beta^3}\sum_{l=1}^{\infty}\frac{e^{\beta l \mu}}{l^3}$$

Now, N is at a maximum when $\mu = 0$. The maximum is when condensation occurs.

So, $\mu \to 0$

$$\Rightarrow N_{3D} = \frac{V}{\pi^2\beta^3}\zeta(3) \approx \frac{V}{\pi^2\beta^3}1.1202$$

gamma function

Solving for the temperture, T_c, required for a BEC to form, we get

$$\boxed{T_c \approx \frac{1}{K}\left[\frac{\pi^2 N_{3D}}{V(1.1202)}\right]^{1/3}}$$

(b) Does Bose-Einstein condensation occur in two-dimensions? justify your answer.

for massless particles, a BEC does occur in 2D. for massive particles, it does not. Since we are dealing with massless particles, the answer is yes.

in 2D, the density of states is

$$D(\epsilon) = 2\pi n \frac{dn}{d\epsilon} = 2\pi\left(\frac{L}{c\hbar\pi}\right)^2 \epsilon$$

so, the total number of particles is

$$(+\ c=1=\hbar) \qquad N_{2D} = \frac{2\pi}{4}\left(\frac{L}{\pi}\right)^2 \int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}\, d\epsilon = \frac{A}{2\pi\beta^2}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l^2}$$

L^2

this factor comes in for the same reason the 1/8 did in the sphere 3-D part. Now the pos. values of n are 1/4 of the area of a circle

when $\mu \to 0$

$$N_{2D} = \frac{A}{2\pi\beta_c^2}\zeta(2) = \frac{A}{2\pi\beta_c^2}\frac{\pi^2}{6}$$

so,

$$\boxed{T_c = \frac{1}{K}\left(\frac{12\ N_{2D}}{A\pi}\right)^{1/2}}$$

(c) what is the highest dimension for which Bose-Einstein condensation does not occur?

for massive particles, 2-D

for massless particles, 1-D