Spring 2005 #14:

photon gas

a) partition function, $\bar{n}_{state}$

$$Z\bar{n}_s = \frac{-1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}$$

$$Z = \sum_R e^{-\beta E_R} = \sum_{R-n_1,n_2,n_3\ldots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

all other terms go away since direvative selects ϵ_s state

easier seen

$$\bar{n}_s = \frac{\sum_{n_s} e^{-\beta n_s \epsilon_s} n_s \cancel{\sum_{n_1 n_2 \ldots}^{(\text{excluding } s)} e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 \ldots)}}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \cancel{\sum_{n_1 n_2 n_3 \ldots}^{(s \text{ excluded})} e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 \ldots)}}}$$

$$= \frac{-1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

$$\sum_{n_s=0}^{\infty} e^{-\beta n_s \epsilon_s} = 1 + e^{-\beta \epsilon_s} + e^{-2\beta \epsilon_s} + \ldots$$

$$= \frac{1}{1 - e^{-\beta \epsilon_s}}$$

$$\bar{n}_s = \frac{\sum_{n_s} e^{-\beta n_s \epsilon_s} n_s}{\sum e^{-\beta n_s \epsilon_s}} = -\frac{1}{\beta}\frac{\partial}{\partial \beta}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

$$\Rightarrow \bar{n}_s = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\frac{1}{1 - e^{-\beta n_s \epsilon_s}}\right)$$

$$= \frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln(1 - e^{-\beta \epsilon_s}) = \frac{e^{-\beta \epsilon_s}}{1 - e^{-\beta \epsilon_s}} = \frac{1}{e^{\beta \epsilon_s} - 1}$$

b) find relation between radiation pressure and mean energy density ($\bar{U}$)

Reif 9.13.20

$$\bar{p} = \sum_s \bar{n}_s \left(-\frac{\partial \epsilon_s}{\partial V}\right)$$ follows from $\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}$ $\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}\frac{\partial \epsilon_s}{\partial V}$, $-\bar{n}_s$

Consider cube $L_x = L_y = L_z$ $V = L^3$

$$\epsilon_s = \hbar\omega = c\underset{\text{wave vector}}{K} = \hbar c\,(K_x^2 + K_y^2 + K_z^2)^{1/2} \qquad K_i = \frac{2\pi}{L_i} n_i$$

$$= \hbar c \left(\frac{2\pi}{L}\right)(n_x^2 + n_y^2 + n_z^2)^{1/2}$$

$$\Rightarrow \epsilon_s = B L^{-1} = B V^{-1/3} \qquad B = \text{constant}$$

$$\frac{\partial \epsilon_s}{\partial V} = -\tfrac{1}{3} B V^{-4/3} = -\tfrac{1}{3}\frac{\epsilon_s}{V}$$

$$\bar{P} = \sum_s \bar{n}_s \left(\tfrac{1}{3}\frac{\epsilon_s}{V}\right) = \frac{1}{3V}\sum_s \bar{n}_s \epsilon_s = \frac{1}{3V}\bar{E} = \frac{1}{3}\bar{u}$$

c) Adiabatic process $\quad dQ = 0 \Rightarrow dE = -p\,dV$

$$p = -\frac{\partial E}{\partial V}$$

$$\Rightarrow \frac{\partial \epsilon_s}{\partial V} = -p_s = -\tfrac{1}{3} B V^{-4/3}$$

$\Rightarrow$ $pV^{4/3} \propto$ constant in general

$$P_0 V_0^{4/3} = P_f V_f^{4/3} \qquad V_f = \frac{1}{8} V_0$$

$$P_f = \bar{P}_0 \left(\frac{V_0}{V_f}\right)^{4/3} = P_0\,(8)^{4/3}$$