Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

Part a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory. (see Yang-Kuo Lim # 5011)

(i) 1st-order

$$E_n^{(1)} = \langle H' \rangle = \langle n | H' | n \rangle = -qE \langle n | x | n \rangle = 0$$

note: we immediately know that this is zero since the only way for it to be non-zero is if the wave functions have opposite parity (since x is odd).

(ii) 2nd-order

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_n^\circ - E_m^\circ}$$

where $E_n^\circ = \omega(n+\frac{1}{2})$ & $E_m^\circ = \omega(m+\frac{1}{2}) \Rightarrow E_n^\circ - E_m^\circ = \omega(n-m)$

and

$$\langle m | H' | n \rangle = -qE \langle m | x | n \rangle = \frac{-qE}{\sqrt{2m\omega}} \langle m | (a+a^+) | n \rangle$$

$$= \frac{-qE}{\sqrt{2m\omega}} \left[\langle m | a | n \rangle + \langle m | a^+ | n \rangle \right]$$

$$\Rightarrow \langle m | H' | n \rangle = \frac{-qE}{\sqrt{2m\omega}} \left[\sqrt{n}\, \delta_{m,n-1} + \sqrt{n+1}\, \delta_{m,n+1} \right]$$

So,

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_n^\circ - E_m^\circ} = \frac{q^2 E^2}{2m\omega} \left[\frac{n}{\omega(n-n+1)} + \frac{n+1}{\omega(n-n-1)} \right]$$

$$\Rightarrow \quad E_n^{(2)} = \frac{q^2 E^2}{2m\omega}\left[\frac{n}{\omega} - \frac{n+1}{\omega}\right]$$

$$\therefore \quad \boxed{E_n^{(2)} = -\frac{q^2 E^2}{2m\omega^2}}$$

part b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

(see Zettili example 9.1 (p473))

the total Hamiltonian is given by

$$H = H_0 + H' = \left(-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right) - qEx$$

let $y = x - \frac{qE}{m\omega^2} \Rightarrow y^2 = x^2 + \frac{q^2E^2}{(m\omega^2)^2} - \frac{2qE}{m\omega^2}x$

so,

$$\frac{1}{2}m\omega^2 y^2 - \frac{q^2E^2}{2m\omega^2} = \frac{1}{2}m\omega x^2 + \frac{q^2E^2}{2m\omega^2} - qEx - \frac{q^2E^2}{2m\omega^2}$$

$$= \frac{1}{2}m\omega^2 x^2 - qEx \quad \checkmark$$

Thus, by this substitution our Hamiltonian is

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 y^2 - \frac{q^2E^2}{2m\omega^2}$$

This is now the Hamiltonian of a harmonic oscillator from which a constant is subtracted. So, the exact eigenstates are

$$\boxed{E_n = \omega\left(n + \frac{1}{2}\right) - \frac{q^2E^2}{2m\omega^2}}$$

→ this agrees exactly with the result of part (a).