Spring 2005 # 2 (p 1 of 3)

Show that in 1-D any attractive potential, no matter how weak, always has at least one bound state. Hint: use variational principle with some appropriate trial wave function such as

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where b is a parameter.

(See Yung-Kuo Lim QM # 8020) ... this solution is a bit suspect.

attractive potential $\Rightarrow V(x) < 0.$

The Hamiltonian is given by

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x) = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 + V(x) - \frac{1}{2}m\omega^2x^2$$

so,

$$E = \langle\psi(x)|H|\psi(x)\rangle = -\frac{1}{2m}\left\langle\frac{d^2}{dx^2}\right\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle + \langle V(x)\rangle - \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$= \langle\psi(x)|H_0|\psi(x)\rangle + \langle\psi(x)|H'|\psi(x)\rangle$$

The reason we wrote the Hamiltonian in this way is because we already know $\langle H_0\rangle$ which is the ground state of the Harmonic oscillator. So, we have

$$E = \frac{\omega}{2} + \langle V(x)\rangle - \frac{1}{2}m\omega\langle x^2\rangle$$

where

$$\langle x^2\rangle = \left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} x^2 e^{-2bx^2}dx = \left(\frac{2b}{\pi}\right)^{1/2}\frac{\sqrt{\pi}}{2(2b)^{3/2}} = \frac{1}{2}\left(\frac{1}{2b}\right) = \frac{1}{4b}$$

so,

$$-\frac{1}{2}m\omega^2\langle x^2\rangle = \frac{-m\omega^2}{8b} = \frac{-\omega}{4}$$

$$\Rightarrow b = \frac{m\omega}{2}$$ to fit the Harmonic oscillator treatment.

So,

$$E = \frac{\omega}{4} + \langle V(x) \rangle = \frac{b}{2m} + \langle V(x) \rangle \qquad (1)$$

Then,

$$\frac{\partial E}{\partial b} = \frac{1}{2m} + \frac{\partial}{\partial b}\left[\sqrt{\frac{2b}{\pi}} \int_{-\infty}^{\infty} e^{-2bx^2} V(x)\, dx\right]$$

$$= \frac{1}{2m} + \underbrace{\frac{1}{2}\sqrt{\frac{2}{\pi b}} \int_{-\infty}^{\infty} e^{-2bx^2} V(x)\, dx}_{\equiv I_1} - \underbrace{2\sqrt{\frac{2b}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-2bx^2} V(x)\, dx}_{\equiv I_2}$$

(since $\frac{\sqrt{b}}{b} = \frac{1}{b}$)

$$\therefore \frac{\partial E}{\partial b} = \frac{1}{2m} + \frac{1}{2b}\langle V(x) \rangle - 2 \langle x^2 V(x) \rangle$$

note: For attractive well, we must have

$$\int_{-\infty}^{\infty} V(x)\, dx \text{ is finite}$$

and

$$\int_{-\infty}^{\infty} x^2 V(x)\, dx \text{ is finite}$$

So,

$$\lim_{b \to 0} I_1 \to \frac{1}{0} \int_{-\infty}^{\infty} V(x)\, dx \to -\infty \qquad (\text{since } V(x) < 0)$$

$$\lim_{b \to 0} I_2 \to 0 \int_{-\infty}^{\infty} x^2 V(x)\, dx \to 0$$

Thus,

$$\boxed{\lim_{b \to 0} \frac{\partial E}{\partial b} \to -\infty}$$

Now,
$$\lim_{b\to\infty} I_1 \to \frac{1}{\infty}\int_{-\infty}^{\infty} 0\cdot V(x)\,dx \to 0$$
$$\lim_{b\to\infty} I_2 \to \infty\int_{-\infty}^{\infty} 0\cdot x^2 V(x)\,dx \to 0$$

goes to zero faster than goes to ∞

Thus,
$$\boxed{\lim_{b\to\infty}\frac{\partial E}{\partial b} \to \frac{1}{2m} > 0}$$

The range of these limits shows us that $\frac{\partial E}{\partial b} = 0$ for some value of b. let us call that value of b, b_0. That is,

$$\frac{\partial E}{\partial b} = 0 = \frac{1}{2m} + \frac{1}{2b_0}\langle V(x)\rangle - 2\langle x^2 V(x)\rangle$$

Solving for $\langle V(x)\rangle$ yields

$$\langle V(x)\rangle = 2b_0\left[2\langle x^2 V(x)\rangle - \frac{1}{2m}\right] \qquad (2)$$

Thus, the energy is (substitute eq (2) into eq (1))

$$E = \langle H(b_0)\rangle = \frac{b_0}{2m} + 4b_0\langle x^2 V(x)\rangle - \frac{b_0}{m}$$

$$\Rightarrow \boxed{E = -\frac{b_0}{2m} + 4b_0\langle x^2 V(x)\rangle < 0} \quad , V(x) < 0$$

$\therefore$ the system has at least one bond state!