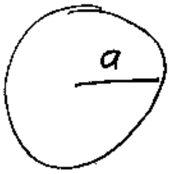
Spring 2005 #3



$V(r) = V_0\, \Theta(a-r)$ with $V_0 = \infty$

Radial wave equation (with $\hbar = 1$)

$$U = r R(r)$$
$$\lambda = 2mV(r)$$

$$U''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)U - \lambda U = 0 \qquad k^2 = 2mE$$

$$0 \qquad r<a$$

$$U''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)U = 0 \qquad r>a$$

$$R_\ell(r) = a_\ell\, j_\ell(kr) + b_\ell\, n_\ell(kr)$$

j_ℓ = spherical Bessel function

n_ℓ = neuman functions

$$R_\ell(a) = 0 = a_\ell\, j_\ell(ka) + b_\ell\, n_\ell(ka)$$

$$a_\ell\, j_\ell(ka) = -b_\ell\, n_\ell(ka)$$

$$\underset{r\to\infty}{j_\ell(kr)} = \frac{\cos\left[kr - (\ell+1)\frac{\pi}{2}\right]}{kr} = \frac{\sin(kr - \frac{\pi\ell}{2})}{kr}$$

$$\underset{r\to\infty}{n_\ell(kr)} = \frac{\sin\left(kr - (\ell+1)\frac{\pi}{2}\right]}{kr} = -\frac{\cos(kr - \frac{\ell\pi}{2})}{kr}$$

$$R_\ell(r) = \frac{a_\ell \sin(kr - \frac{\pi\ell}{2})}{kr} - \frac{b_\ell \cos(kr - \frac{\ell\pi}{2})}{kr}$$

But then we know that another form of the solution is

$$R_\ell(r) = \frac{e^{i\delta_\ell}}{kr}\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell\right) \quad \text{with } \tan\delta_\ell = \frac{-b_\ell}{a_\ell} = \frac{j_\ell(ka)}{n_\ell(ka)}$$

arbitrary phase constant.

This works since $\sin(kr - \frac{\pi\ell}{2} + \delta_\ell) = \sin(kr - \frac{\pi\ell}{2})\cos\delta_\ell + \cos(kr - \frac{\pi\ell}{2})\sin\delta_\ell$

Then by comparison $\cos\delta_\ell = a_\ell$

$\sin\delta_\ell = -b_\ell$

But now, we want $\ell = 0$ case.

$$\Rightarrow \tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = -\tan(ka). \quad \Rightarrow \delta_0 = -ka$$

for small k

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0 \, ka \approx \frac{4\pi(ka)^2}{k^2} = 4\pi a^2$$

small angle approximation.