Spring 2005 #3 (p 1 of 2)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is ~~zero~~ zero outside the sphere, infinite inside. The wave function must vanish at $r=a$.

What is the S-wave ($\ell=0$) phase shift as a function of the incident energy or momentum? what is the total cross section in the limit of zero incident kinetic energy?

see spring 1999 #13. & Abers p 283, 284

The potential we are given has the form

$$V(r) = V_0 \, \Theta(a-r) \quad \text{in the limit } V_0 \to \infty$$

we know that the radial wave equation is

$$\frac{d^2u}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u - \lambda u = 0 \qquad , \; k^2 = 2mE \; \& \; \lambda = 2mV(r)$$

where $u = rR(r)$.

substituting in for the value of the potential yields

$$\frac{d^2u}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]u = 0 \qquad r > a$$

there is no equation for $r < a$ since it is an impenetrable sphere. The solutions to this D.E. are

$$R_\ell(r) = a_\ell \, j_\ell(kr) + b_\ell \, n_\ell(kr) \qquad (1)$$

we are told that

$$R_\ell(r=a) = 0 = a_\ell \, j_\ell(ka) + b_\ell \, n_\ell(ka)$$

$$\Rightarrow \boxed{\frac{-b_\ell}{a_\ell} = \frac{j_\ell(ka)}{n_\ell(ka)} \equiv \tan\delta_\ell} \qquad (2)$$

as $r \to \infty$, our solution given by eq (1) becomes

$$R_\ell(r) = \frac{a_\ell \sin(kr - \frac{\pi \ell}{2})}{kr} - \frac{b_\ell \cos(kr - \pi\ell/2)}{kr}$$

we know from solutions to D.E.s that we can also write this solution as

$$R_\ell(r) = \frac{e^{i\delta_\ell}}{kr} \sin\left(kr - \frac{\pi \ell}{2} + \delta_\ell\right)$$

where $e^{i\delta_\ell}$ is a phase shift to make sure that the outgoing wave is only due to the scattering and not the plane wave.

Now, we want to find the s-wave ($\ell = 0$) phase shift. From eq (2) this becomes

$$\tan \delta_0 = \frac{j_0(ka)}{n_0(ka)} = \frac{\frac{\sin ka}{ka}}{-\frac{\cos ka}{ka}} = -\tan ka$$

$$\therefore \boxed{\delta_0 = -ka} \quad , \; k = \sqrt{2mE}$$

Now, we want the total cross section. From Abers eq 8.121 we know that in the limit as $k \to 0$, σ_{tot} is given by

$$\sigma_{tot} = \frac{4\pi}{k^2} \sin^2 \delta_0$$

since $\delta_0 \ll 1$ for our case (since $k \to 0$) we have

$$\boxed{\sigma_{tot} = \frac{4\pi}{k^2} \delta_0^2 = 4\pi a^2}$$