A electron is at rest in a constant magnetic field pointing in the z-direction. The Hamiltonian is then

$$H = -\vec{\mu}\cdot\vec{B} = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B}$$

where $\vec{B} = B_0\hat{z}$.
Let $|\psi_\pm\rangle$ be the eigenstates of S_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

part a) What are the eigenstates of the Hamiltonian, and what is the energy difference between them?

(See spring 1999 #11)

$\vec{B} = B_0\hat{z}$ $\qquad H = \frac{g\mu_0 B_0}{2}\sigma_z = \frac{g\mu_0 B_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Since this is a diagonal matrix, the eigenvalues are just the elements along the diagonal. That is,

$$\lambda = \pm\frac{g\mu_0 B_0}{2}$$

So the eigenstates are

$|\lambda = -g\frac{\mu_0 B_0}{2}\rangle$: $\begin{pmatrix} g\mu_0 B_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow |\lambda_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |\psi_-\rangle$

$|\lambda = +g\frac{\mu_0 B_0}{2}\rangle$: $\begin{pmatrix} 0 & 0 \\ 0 & -g\mu_0 B_0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow |\lambda_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\psi_+\rangle$

the energy difference is $\boxed{\Delta E = g\mu_0 B_0}$

(b) At time $t=0$ the electron is in an eigenstate of S_x with eigenvalue $\hbar/2$. Calculate $|\psi(t)\rangle$ for any t.

$S_x = \frac{\sigma_x}{2}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_{\sigma_x} = \pm 1$; $|\lambda = +1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(see S99 #11 for details)

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|\psi_+\rangle + \frac{1}{\sqrt{2}}|\psi_-\rangle$$

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-iH_+t}\,|\psi_+\rangle + \frac{1}{\sqrt{2}} e^{-iH_-t}\,|\psi_-\rangle$$

$$\therefore \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\, e^{-ig\frac{\mu_0 B_0}{2}}\,|\psi_+\rangle + \frac{1}{\sqrt{2}}\, e^{+ig\frac{\mu_0 B_0}{2}}\,|\psi_-\rangle}$$

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t? let $\alpha = g\frac{\mu_0 B_0}{2}$

(i) $\langle S_x\rangle$

$$\langle S_x\rangle = \langle\psi(t)|S_x|\psi(t)\rangle = \frac{1}{2}\langle\sigma_x\rangle = \frac{1}{4}\left(e^{i\alpha t}\;\; e^{-i\alpha t}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix}$$

$$= \frac{1}{4}\left(e^{i\alpha t}\;\; e^{-i\alpha t}\right)\begin{pmatrix}e^{i\alpha t}\\ e^{-i\alpha t}\end{pmatrix} = \frac{1}{4}\left(e^{i2\alpha t} + e^{-i2\alpha t}\right)$$

$$\therefore \boxed{\langle S_x\rangle = \frac{1}{2}\cos(2\alpha t) = \frac{1}{2}\cos(g\mu_0 B_0 t)}$$

(ii) $\langle S_y\rangle$ -

$$\langle S_y\rangle = \frac{1}{2}\langle\sigma_y\rangle = \frac{1}{4}\left(e^{i\alpha t}\;\; e^{-i\alpha t}\right)\begin{pmatrix}0 & -i\\ +i & 0\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix}$$

$$\Rightarrow \boxed{\langle S_y\rangle = \frac{1}{2}\sin(g\mu_0 B_0 t)}$$

(iii) $\langle S_z\rangle$

$$\boxed{\langle S_z\rangle = 0}\quad , \langle S_z\rangle = \frac{1}{4}\left(e^{i\alpha t}\;\; e^{-i\alpha t}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix} = 0$$