Spring 2005 #5 natural units $\hbar = c = 1$

$$\psi(r,\theta,\phi) = N\, R_{21}(r)\left[2i\, Y_1^{-1} + (2+i)\, Y_1^{0} + 3i\, Y_1^{1}\right]$$

Y_l^m, R_{nl} Remember Y_l^m's are normalized

$\int Y_l^{*m} Y_l^m \sin\theta\, d\theta\, d\phi = 1$

a) Find N

$$1 = N^2 \int \psi^* \psi \, d^3 r = N^2 \Big[\int_0^{\infty} R_{21}^2(r)\,(-2i)(2i)\, Y_l^{*-1} Y_l\, r^2 \sin\theta\, d\theta\, d\phi$$

$$+ i5 \int_0^{\infty} R_{21}^2 r^2 dr + 9 \int_0^{\infty} R_{21}^2 r^2 dr \Big]$$

$$= 18 N^2 \int_0^{\infty} |R_{21}|^2 r^2 dr$$ But Radial wave equations are also normalized.

$18N^2 = 1 \quad N^2 = \frac{1}{18} \quad N = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$

b) $L_z Y_l^m = m\, Y_l^m$

$$\langle \psi | L_z | \psi \rangle = N^2 \int |R_{21}|^2 \left[(4)\, Y_1^{*-1} L_z Y_1^{-1} + 5\, Y_1^{*0} L_z Y_1^{0} + 9\, Y_1^{1*} L_z Y_1^{1}\right] d^3 r$$

$$= N^2 \int |R_{21}|^2 \left[4\cdot(-1) + 5\cdot(0) + 9\cdot(1)\right] r^2\, dr$$

$$= N^2 5 \int |R_{21}|^2 r^2 dr \qquad -4+9=5$$

$$= N^2 5$$

$$= \frac{5}{18}$$

c) $L^2 Y_\ell^m = \ell(\ell+1) Y_\ell^m$

$$\langle\psi|L^2|\psi\rangle = N^2 \int |R_{21}|^2 r^2 dr \left[\underbrace{4(1(1+1)) + 5(1(1+1)) + 9(1(1+1))}_{8 + 10 + 18 = 36}\right]$$

$$= N^2 36$$

$$= \frac{36}{18} = 2$$

d) $V(r) = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$ $\qquad r\frac{\partial}{\partial r}\left(\frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}\right) = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} = -V$

$$2\langle T\rangle = \langle r\frac{\partial}{\partial r}V(r)\rangle = \langle -V\rangle$$

$$\langle T\rangle = \frac{-1}{2}\langle V\rangle$$

$$\langle H\rangle = \langle T\rangle + \langle V\rangle = \frac{-1}{2}\langle V\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle$$

$$\langle H\rangle = -\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} = \frac{1}{2}\langle V\rangle$$

Seting $\hbar = \hbar$ for this section.

$$\langle V\rangle = \frac{-m_e}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{n^2}$$

$$\Rightarrow \langle T\rangle = \frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{n^2}$$