Spring 2005 #5 (p 1 of 2)

An electron moves in a hydrogen atom potential - ignoring spin and relativity - in a state $|\psi\rangle$ that has the wave function

$$\psi(r,\theta,\phi) = N R_{21}(r)\left[2i\, Y_1^{-1}(\theta,\phi) + (2+i)\, Y_1^{0}(\theta,\phi) + 3i\, Y_1^{1}(\theta,\phi)\right]$$

where the $Y_\ell^m(\theta,\phi)$ are the spherical harmonics, $R_{n\ell}(r)$ are the normalized hydrogen atom wave functions, and N is a positive real number.

(See Abers Final from 221A #2)

we can re-write this wave function for simplicity. That is,

$$|\psi\rangle = N\left[c_{m=-1}\,|2,1,-1\rangle + c_{m=0}\,|2,1,0\rangle + c_{m=1}\,|2,1,1\rangle\right]$$

where $c_{m=-1} = 2i$, $c_{m=0} = (2+i)$, and $c_{m=1} = 3i$

(a) Calculate N.

$$1 = |N|^2\left[\,|2i|^2 + |2+i|^2 + |3i|^2\,\right] = |N|^2\left[4+5+9\right]$$

$$\therefore \boxed{N = \frac{1}{\sqrt{18}}}$$

(b) what is the expectation value of L_z?

$$L_z\,|n,\ell,m\rangle = m\,|n,\ell,m\rangle$$

so

$$\langle L_z\rangle = |N|^2\left[\,|2i|^2\langle 2,1,-1|L_z|2,1,-1\rangle + |2+i|^2\langle 2,1,0|L_z|2,1,0\rangle + \right.$$

$$\left. + |3i|^2\langle 2,1,1|L_z|2,1,1\rangle\right]$$

$$= \frac{1}{18}\left[4(-1) + 5(0) + 9(1)\right] = \boxed{\frac{5}{18}}$$

(c) what is the expectation value of L^2?

$$L^2 |n,\ell,m\rangle = \ell(\ell+1)\, |n,\ell,m\rangle$$

so,

$$\langle L^2 \rangle = |N|^2 \Big[|2i|^2 \langle 2,1,-1| L^2 |2,1,-1\rangle + |2+i|^2 \langle 2,1,0| L^2 |2,1,0\rangle + $$
$$+ |3i|^2 \langle 2,1,1| L^2 |2,1,1\rangle \Big]$$

$$= \frac{1}{18} \left[4 \cdot 1(1+1) + 5 \cdot 1(1+1) + 9 \cdot 1(1+1) \right]$$

$$= \frac{1}{9} \cdot 18 = \boxed{2}$$

(d) what is the expectation value of the kinetic energy in terms of $\hbar$, c, the fine structure constant α, and the electron mass m?

From the virial theorem we know that

$$\langle T \rangle = -E_n \quad \leftarrow \text{ see Griffiths' QM 4.41}$$

the energy levels for a hydrogen atom are (in natural units)

$$E_n = -\frac{\alpha^2 m}{2n^2}$$

so, for $n = 2$

$$\boxed{\langle T \rangle = +\frac{\alpha^2 m}{8}}$$