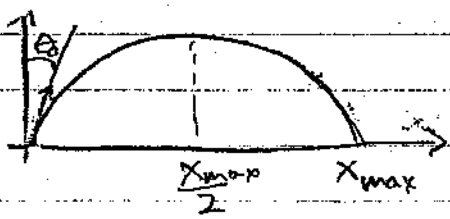
Spring 1995 #1

Consider the propagation of electromagnetic waves in a nonuniform medium with refractive index $n^2 = 1 - \frac{\omega_p^2}{\omega^2}$ where

$$\frac{\omega_p^2}{\omega^2} = \frac{y}{h} \qquad \omega, h = \text{constant}$$

At the origin $(x=y=0)$ the plane wave propagates at an angle Θ_0 with respect to the y-axis. Using Snell's Law, find the maximum height y_{max} which the wave assumes. Calculate the ray trajectory $y(x)$ and find the maximum horizontal distance X_{max} where the wave reaches the original height $(y=0)$.



→ the index of refraction is given by $n = \sqrt{1 - y/h}$

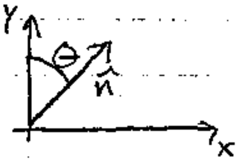
using Snell's Law $\quad n_0 \sin\Theta_0 = n_{y_{max}} \sin(\pi/2)$

$$\sin\Theta_0 = \sqrt{1 - \frac{y_{max}}{h}}$$

$$1 - \frac{y_{max}}{h} = \sin^2\Theta_0$$

$$\boxed{y_{max} = h(1 - \sin^2\Theta_0) = h(\cos^2\Theta_0)}$$

to find trajectory, consider a unit vector pointing in the direction of propagation



$dy = \cos\Theta \qquad dx = \sin\Theta$

thus $\frac{dy}{dx} = \frac{\cos\Theta}{\sin\Theta}$

solve for $\sin\Theta$ using Snell's Law $\quad n_0 \sin\Theta_0 = n \sin\Theta$

$$\sin\theta_0 = \sqrt{1-y/h}\,\sin\theta \quad \Rightarrow \sin\theta = \frac{\sin\theta_0}{\sqrt{1-y/h}}$$

we know that $\sin\theta_0 = \sqrt{1-\frac{y_{max}}{h}}$

thus $\sin\theta = \sqrt{\frac{1-\frac{y_{max}}{h}}{1-y/h}} = \sqrt{\frac{h-y_{max}}{h-y}}$

now use $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{\frac{\cancel{h}-y-\cancel{h}+y_{max}}{h-y}} = \sqrt{\frac{y_{max}-y}{h-y}}$

thus $\frac{dy}{dx} = \sqrt{\frac{y_{max}-y}{h-y_{max}}}$

$$\frac{dy}{\sqrt{y_{max}-y}} = \frac{dx}{\sqrt{h-y_{max}}} \Rightarrow \text{let } u = y_{max}-y, \quad du = -dy$$

(1) $-\int u^{-1/2}\,du = -2\sqrt{y_{max}-y} = \frac{x+c}{\sqrt{h-y_{max}}}$

$$y_{max}-y = \frac{1}{4}\frac{(x+c)^2}{(h-y_{max})}$$

$$y = y_{max} - \frac{1}{4}\frac{(x+c)^2}{(h-y_{max})}$$

initial condition

$$0 = y_{max} - \frac{1}{4}\frac{c^2}{(h-y_{max})} \Rightarrow \sqrt{4y_{max}(h-y_{max})} = c$$

so $\boxed{y(x) = y_{max} - \frac{1}{4}\frac{\left(x+\sqrt{4y_{max}(h-y_{max})}\right)^2}{(h-y_{max})}}$

to find range by integrating (1) from $y: 0 \to y_{max}$, $x: 0 \to \frac{1}{2}x_{max}$

$$-2\left[0-\sqrt{y_{max}}\right] = \frac{1}{2}\frac{x_{max}}{\sqrt{h-y_{max}}}$$

$$\boxed{x_{max} = 4\sqrt{y_{max}(h-y_{max})}}$$