Fall 1996 #1

A ~~solid~~ metal sphere has a radius R and a charge Q.

(i) Compute the electric part of the Maxwell Stress Tensor:

$$T_{ij}(\vec{r}) = \frac{1}{4\pi}\left\{ E_i E_j - \frac{1}{2}\vec{E}^2 \delta_{ij} \right\}$$

both inside and outside the sphere

(see Griffith's pgs 326-330)

→ Outside the sphere, $\vec{E} = \frac{Q}{r^2}\hat{r}$

Recall in Cartesian coordinates:

$$\hat{r} = \sin\Theta\cos\phi\,\hat{i} + \sin\Theta\sin\phi\,\hat{j} + \cos\Theta\,\hat{k}$$

Now we can find T_{xx}, T_{yy}, T_{zz}, $T_{xy}=T_{yx}$, $T_{xz}=T_{zx}$, $T_{yz}=T_{zy}$

$$\left.\begin{aligned}
T_{xx} &= \frac{1}{4\pi}\left[\frac{Q^2}{r^4}\sin^2\Theta\cos^2\phi - \frac{1}{2}\frac{Q^2}{r^4}\right] \\
T_{yy} &= \frac{1}{4\pi}\left[\frac{Q^2}{r^4}\sin^2\Theta\sin^2\phi - \frac{1}{2}\frac{Q^2}{r^4}\right] \\
T_{zz} &= \frac{1}{4\pi}\left[\frac{Q^2}{r^4}\cos^2\Theta - \frac{1}{2}\frac{Q^2}{r^4}\right] \\
T_{xy} = T_{yx} &= \frac{1}{4\pi}\frac{Q^2}{r^4}\sin^2\Theta\sin\phi\cos\phi \\
T_{xz} = T_{zx} &= \frac{1}{4\pi}\frac{Q^2}{r^4}\sin\Theta\cos\Theta\cos\phi \\
T_{yz} = T_{zy} &= \frac{1}{4\pi}\frac{Q^2}{r^4}\sin\Theta\cos\Theta\sin\phi
\end{aligned}\right\}\ \text{Components of Maxwell stress tensor outside of sphere}$$

Inside the sphere $\vec{E}$ can be found using Gauss's Law: $\nabla\cdot E = 4\pi\rho$

$$\oint E\cdot da = E(4\pi r^2) = 4\pi\frac{(\frac{4}{3})\pi r^3}{(\frac{4}{3})\pi R^3}Q$$

$$E = Q\frac{r}{R^3}\hat{r}$$

$$T_{xx} = \frac{1}{4\pi}\left[\frac{Q^2 r^2}{R^6}\sin^2\Theta\cos^2\phi - \frac{1}{2}\frac{Q^2 r^2}{R^6}\right]$$

$$T_{yy} = \frac{1}{4\pi}\left[\frac{Q^2 r^2}{R^6}\sin^2\Theta\sin^2\phi - \frac{1}{2}\frac{Q^2 r^2}{R^6}\right]$$

$$T_{zz} = \frac{1}{4\pi}\left[\frac{Q^2 r^2}{R^6}\cos^2\Theta - \frac{1}{2}\frac{Q^2 r^2}{R^6}\right]$$

$$T_{xy} = T_{yx} = \frac{1}{4\pi} \frac{Q^2 r^2}{R^6} \sin^2\Theta \sin\phi \cos\phi$$

$$T_{xz} = T_{zx} = \frac{1}{4\pi} \frac{Q^2 r^2}{R^6} \sin\Theta \cos\Theta \cos\phi$$

$$T_{yz} = T_{zy} = \frac{1}{4\pi} \frac{Q^2 r^2}{R^6} \sin\Theta \cos\Theta \sin\phi$$

(ii) We now cut the sphere in half along the x-y plane. The two hemispheres repel each other with a force $\vec{F}$. Compute the magnitude and direction of the force $\vec{F}$ on the upper hemisphere using (i).

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a}$$

1st consider 'bowl' part of upper hemisphere: The force will point along $\hat{z}$ direction, so we need to calculate $(\overleftrightarrow{T} \cdot d\vec{a})_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$ where $d\vec{a} = R^2 \sin\Theta \, d\Theta \, d\phi \, (\sin\Theta\cos\phi\,\hat{i} + \sin\Theta\sin\phi\,\hat{j} + \cos\Theta\,\hat{k})$

$$(\overleftrightarrow{T} \cdot d\vec{a})_z = \left(\frac{1}{4\pi}\frac{Q^2}{R^2}\sin^2\Theta\cos\Theta\cos^2\phi + \frac{1}{4\pi}\frac{Q^2}{R^2}\sin^2\Theta\sin^2\phi\cos\Theta + \frac{1}{4\pi}\frac{Q^2}{R^2}\left[\cos^2\Theta - \frac{1}{2}\right]\cos\Theta\right)\sin\ldots$$

simplification yields $(\overleftrightarrow{T} \cdot d\vec{a})_z = \frac{1}{8\pi}\frac{Q^2}{R^2} \sin\Theta\cos\Theta \, d\Theta \, d\phi$

$$\text{Force on 'bowl'} = \frac{1}{4}\frac{Q^2}{R^2}\int_0^{\pi/2} \sin\Theta\cos\Theta \, d\Theta = \frac{1}{8}\frac{Q^2}{R^2}\hat{z}$$

Now find force on disk:

$$(\overleftrightarrow{T} \cdot d\vec{a})_z = T_{zz}\, da_z = \frac{1}{4\pi}\left[\frac{Q^2 r^2}{R^6}\cos^2\Theta - \frac{1}{2}\frac{Q^2 r^2}{R^6}\right]\underbrace{(-r\,dr\,d\phi)}_{=d\vec{a}}$$

for disk $\Theta = \pi/2$

$$(\overleftrightarrow{T} \cdot d\vec{a})_z = +\frac{1}{8\pi}\frac{Q^2}{R^6}\int_0^{2\pi}\int_0^R r^3\,dr\,d\phi = +\frac{1}{4}\frac{Q^2}{R^6}\frac{R^4}{4} = \frac{Q^2}{16R^2}\hat{z}$$

Thus total force $= \frac{Q^2}{R^2}\left(\frac{1}{8} + \frac{1}{16}\right)\hat{z}$

$$\boxed{\vec{F} = \frac{3Q^2}{16R^2}\hat{z}}$$