Fall 1996 #11

An electron with charge e and mass m is confined to move on a circle of radius r. It is perturbed by a uniform electric field $\vec{F}$ parallel to one of the diameters of the circle.

a) Find the unperturbed energy levels.

$$H = \frac{p_\phi^2}{2mr^2} = \frac{-\hbar^2}{2mr^2}\frac{d^2}{d\phi^2}$$

$$-\frac{\hbar^2}{2mr^2}\frac{d^2\psi}{d\phi^2} = E\psi \quad \Rightarrow \quad \frac{d^2\psi}{d\phi^2} = \frac{-2mEr^2}{\hbar^2}\psi = -n^2\psi$$

$$n = \sqrt{\frac{2mEr^2}{\hbar^2}}$$

$$\psi = \frac{e^{in\phi}}{\sqrt{2\pi}} \quad \rightarrow \psi \text{ must be periodic, with } \psi(\phi+2\pi) = \psi(\phi)$$

thus n must be an integer

solving for E yields: $\boxed{E = \frac{\hbar^2}{2mr^2}n^2}$ $\quad n = 0, \pm 1, \pm 2 \ldots$

b) Find the shift to first order in F.

→ perturbing Hamiltonian given by $H' = e\Phi = eFx = eFr\cos\phi$

the 1st order shift is $\langle \psi_n^\circ | H' | \psi_n^\circ \rangle = \Delta E^{(1)}$

$$\Delta E^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}\, eFr\cos\phi\, e^{in\phi}\, d\phi = eFr\int_0^{2\pi}\cos\phi\, d\phi = 0$$

$\boxed{\Delta E^{(1)} = 0}$ ← there is no shift to first order in F

c) Find the second order shift. Notice in particular the special care needed for the first excited state.

2nd order shift is $\Delta E^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^\circ | H' | \psi_n^\circ \rangle|^2}{E_n^\circ - E_m^\circ}$

calculate $\langle \psi_m^\circ | H' | \psi_n^\circ \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\, eFr\cos\phi\, e^{in\phi}\, d\phi$

write $\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$ $\Rightarrow$

$$\langle \psi_m^0 | H' | \psi_n^0 \rangle = \frac{eFr}{4\pi} \int_0^{2\pi} e^{-im\phi + in\phi} \left(e^{i\phi} + e^{-i\phi} \right) d\phi$$

$$= \frac{eFr}{4\pi} \int_0^{2\pi} \left(e^{i\phi(-m+n+1)} + e^{i\phi(-m+n-1)} \right) d\phi$$

$$= \begin{cases} \frac{eFr}{4\pi}(2\pi) & \text{for } m = n+1,\ m = n-1 \\ 0 & \text{otherwise} \end{cases}$$

thus $\Delta E^{(2)} = \dfrac{e^2F^2r^2}{4\left(\frac{\hbar^2}{2mr^2}\right)\left[n^2-(n+1)^2\right]} + \dfrac{e^2F^2r^2}{4\left(\frac{\hbar^2}{2mr^2}\right)\left[n^2-(n-1)^2\right]}$

$$\Delta E^{(2)} = \frac{2e^2F^2r^4m}{4\hbar^2}\left[\left(\frac{1}{\cancel{n^2}-\cancel{n^2}-2n-1}\right) + \left(\frac{1}{\cancel{n^2}-\cancel{n^2}+2n-1}\right)\right]$$

$$\Delta E^{(2)} = \frac{2e^2F^2r^4\pi^2m}{4\hbar^2}\left[\frac{\cancel{2n}-1}{(-2n-1)(2n-1)} + \frac{-\cancel{2n}-1}{(-2n-1)(2n-1)}\right]$$

$$\Delta E^{(2)} = \frac{2e^2F^2r^4\pi^2m}{4\hbar^2}\left(\frac{+2}{+4n^2-1}\right)$$

$$\boxed{\Delta E^{(2)} = \frac{e^2F^2r^4\pi^2m}{\hbar^2(4n^2-1)}}$$

mass