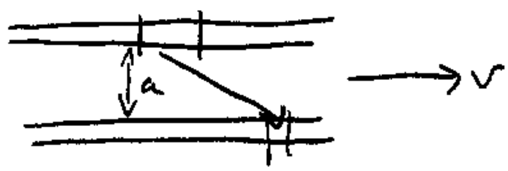
Fall 1996 #3 (p 1 of 2)

Two thin, parallel infinitely long, non-conducting rods, a distance a apart, with identical constant charge density λ per unit length in their rest frame, move with a velocity v, not necessarily small compared to the speed of light. Calculate the force per unit length between them in a frame of reference that's at rest, and in a frame of reference moving with the rods, and compare the results. (See U. of Chicago Grad. problems w/ solutions E & M #42)



In the rest frame, the Force per unit length F is given by

$$\vec{F} = \lambda \vec{E}$$

where $\vec{E}$ is the electric field at one wire produced by the other. We can find $\vec{E}$ from Gauss' Law,

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enclosed} = 4\pi \lambda L$$

$$\Rightarrow E = \frac{2\lambda}{a}$$

So, the force is

$$F = +\frac{2\lambda^2}{a}, \text{ where the force is repulsive}$$

In a frame in which the rods are seen to move with velocity v, there is a magnetic field $\vec{B} = v \times \frac{\vec{E}'}{c}$ in addition to the electric field $\vec{E}'$. The total force per unit length F' is then

$$\vec{F}' = \lambda' \left(\vec{E}' + \frac{\vec{v} \times \vec{B}'}{c}\right) = \lambda' \left(1 - \frac{v^2}{c^2}\right) \vec{E}'$$

where $\vec{E}' = 2\lambda'/a$ and λ' is the charge as seen in the new frame. So

$$\lambda' = \gamma \lambda$$

Thus,

$$F' = \frac{2(\lambda')^2(1 - v^2/c^2)}{a} = \frac{2(\gamma\lambda)^2 \; 1/\gamma^2}{a} = \frac{2\lambda^2}{a} = F$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

The fact that $F' = F$ may be seen easily by an alternate argument. If in its rest frame, one of the rods is allowed to move under the action of the force FL on it, it would gain momentum $dp = FL\,dt$, while in the frame in which the rods move, the gain is $dp' = F'L'dt$. But $dp = dp'$ because momenta normal to the direction of a Lorentz transformation are invariant under such a transformation, and $dt' = \gamma dt$, hence $LF = \gamma F'L'$. In addition $L' = L/\gamma$, due to Lorentz contraction; hence $F = F'$.