Fall 1996 #9

Consider a spin ½ particle. Show that in the space of the states of a given orbital angular momentum ℓ, the operators

$$\Lambda_+ = \frac{\ell+1+\vec{L}\cdot\vec{\sigma}}{2\ell+1} \qquad \Lambda_- = \frac{\ell-\vec{L}\cdot\vec{\sigma}}{2\ell+1}$$

are projection operators onto the states of total angular momentum $j=\ell+\frac{1}{2}$ and $j=\ell-\frac{1}{2}$ respectively.

→ Note that $\vec{J}=\vec{L}+\vec{S}$

Thus $J^2=(\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S})$

$$J^2 = L^2+2\vec{L}\cdot\vec{S}+S^2$$

recall $\vec{S}=\frac{\hbar}{2}\vec{\sigma} = \frac{\vec{\sigma}}{2} \quad (\hbar=1)$

$$J^2 = L^2 + \vec{L}\cdot\vec{\sigma} + S^2$$

thus

$$\vec{L}\cdot\vec{\sigma} = J^2-L^2-S^2$$

Let Λ_+ act on $|j=\ell+\frac{1}{2}\rangle$

$$\Lambda_+|j=\ell+\tfrac{1}{2}\rangle = \frac{\ell+1+\vec{L}\cdot\vec{\sigma}}{2\ell+1}|j=\ell+\tfrac{1}{2}\rangle$$

$$= \frac{\ell+1+(\ell+\frac{1}{2})(\ell+\frac{3}{2})-\ell(\ell+1)-\frac{3}{4}}{2\ell+1}|j=\ell+\tfrac{1}{2}\rangle$$

$$= \frac{\ell+1+\cancel{\ell^2}+2\ell+\cancel{\tfrac{3}{4}}-\cancel{\ell^2}-\cancel{\ell}-\cancel{\tfrac{3}{4}}}{2\ell+1}|j=\ell+\tfrac{1}{2}\rangle$$

$$\Lambda_+|j=\ell+\tfrac{1}{2}\rangle = |j=\ell+\tfrac{1}{2}\rangle$$

let Λ_+ act on $|j=\ell-\frac{1}{2}\rangle$

$$\Lambda_+|j=\ell-\tfrac{1}{2}\rangle = \frac{\ell+1+\vec{L}\cdot\vec{\sigma}}{2\ell+1}|j=\ell-\tfrac{1}{2}\rangle$$

$$= \frac{\ell+1+(\ell-\frac{1}{2})(\ell+\frac{1}{2})-\ell(\ell+1)-\frac{3}{4}}{2\ell+1}|j=\ell-\tfrac{1}{2}\rangle$$

$$= \frac{\ell+1+\cancel{\ell^2}-\frac{1}{4}-\cancel{\ell^2}-\ell-\frac{3}{4}}{2\ell+1}|j=\ell-\tfrac{1}{2}\rangle$$

$$= 0\,|j=\ell-\tfrac{1}{2}\rangle$$

Thus $\Lambda_+ |j=\ell+1/2\rangle = |j=\ell+1/2\rangle$

$\Lambda_+ |j=\ell-1/2\rangle = 0$

So Λ_+ acts as a projection operator on state $|j=\ell+1/2\rangle$

Let Λ_- act on $|j=\ell+1/2\rangle$

$$\Lambda_- |j=\ell+1/2\rangle = \frac{\ell-\vec{L}\cdot\vec{\sigma}}{2\ell+1}|j=\ell+1/2\rangle$$

$$= \frac{\ell-(\ell+1/2)(\ell+3/2)+\ell(\ell+1)+3/4}{2\ell+1}|j=\ell+1/2\rangle$$

$$= \frac{\ell-\ell^2-2\ell-3/4+\ell^2+\ell+3/4}{2\ell+1}|j=\ell+1/2\rangle$$

$$\Lambda_- |j=\ell+1/2\rangle = 0$$

Let Λ_- act on $|j=\ell-1/2\rangle$

$$\Lambda_- |j=\ell-1/2\rangle = \frac{\ell-(\ell-1/2)(\ell+1/2)+\ell(\ell+1)+3/4}{2\ell+1}|j=\ell-1/2\rangle$$

$$= \frac{\ell-\ell^2+1/4+\ell^2+\ell+3/4}{2\ell+1}|j=\ell-1/2\rangle$$

$$\Lambda_- |j=\ell-1/2\rangle = |j=\ell-1/2\rangle$$

Thus $\Lambda_- |j=\ell+1/2\rangle = 0$

$\Lambda_- |j=\ell-1/2\rangle = |j=\ell-1/2\rangle$

So Λ_- acts as a projection operator on $|j=\ell-1/2\rangle$