Fall 1997 #1

p109

There is a constant magnetic induction $\vec{B} = B\hat{z}$ throughout a certain region of space.

a) Show that the vector potential there can be written in the form $\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r})$.

→ Calculate $\vec{A} = \frac{1}{2}(B\hat{z} \times (x\hat{x} + y\hat{y} + z\hat{z}))$

$$= \tfrac{1}{2} Bx(\hat{z}\times\hat{x}) + \tfrac{1}{2} By(\hat{z}\times\hat{y}) + \tfrac{1}{2} Bz(\hat{z}\times\hat{z})$$

$$= \tfrac{1}{2}Bx\,\hat{y} + \tfrac{1}{2}By(-\hat{x}) + 0$$

$$\vec{A} = -\tfrac{1}{2}By\,\hat{x} + \tfrac{1}{2}Bx\,\hat{y}$$

verify $\nabla\times\vec{A} = \vec{B}$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{1}{2}By & \frac{1}{2}Bx & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}\left(\tfrac{1}{2}B + \tfrac{1}{2}B\right)$$

thus $\nabla\times\vec{A} = \vec{B}$

so one vector potential for $\vec{B} = B\hat{z}$ is $\boxed{\vec{A} = -\frac{1}{2}By\,\hat{x} + \frac{1}{2}Bx\,\hat{y}}$

One can obtain a different vector potential by adding the gradient of any scalar function to $\vec{A}$

$$\vec{A}' = \vec{A} + \nabla\Lambda \qquad \text{for } \Lambda = \text{scalar function}$$

Since the curl of a gradient is zero, the magnetic induction is unaffected

b) A constant magnetic induction $\vec{B} = B\hat{z}$ is present everywhere except inside a long cylinder of radius a with axis parallel to $\hat{z}$. $\vec{B} = 0$ inside the cylinder. Find the vector potential which is valid $\vec{A}(\vec{r})$ everywhere.

→ (refer to Griffiths pgs 230-231)

⇒

$C \to C(1+x)$

$= \varepsilon C$

Notice that:

$$\oint \vec{A}\cdot d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi$$

where Φ is the flux through the loop in question.

This is similar to Ampere's law in integral form:

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$

So with the substitutions $\vec{B} \to \vec{A}$, $\frac{4\pi}{c} I_{enc} \to \Phi$, we obtain an 'Ampere's Law' for vector potentials.

Thus, for $r < a$, $\Phi = 0$ so we get $\vec{A} = 0$ $(r<a)$

For $r > a$, $\Phi = \pi B(r^2 - a^2)$, and $\oint \vec{A}\cdot d\vec{l} = 2\pi r A$

so

$$A = \frac{\pi B(r^2-a^2)}{2\pi r}$$

since $\vec{A}$ mimics $\vec{B}$ from ampere's law, one can use the right rule to find that $\vec{A}$ points in the $\hat{\phi}$ direction.

So,

$$\boxed{\vec{A} = \begin{cases} 0 & (r<a) \\ \frac{B(r^2-a^2)}{2r}\hat{\phi} & (r>a) \end{cases}}$$

verify $\nabla\times\vec{A} = \vec{B}$ (see Arfken pg 112)

$$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ 0 & \frac{B}{2}(r^2-a^2) & 0 \end{vmatrix}$$

$$\nabla\times\vec{A} = \frac{1}{r}\hat{r}(0) - \frac{1}{r}r\hat{\phi}(0) + \frac{1}{r}\hat{z}\left(\frac{\partial}{\partial r}\left(\frac{B}{2}(r^2-a^2)\right)\right)$$

$$\nabla\times\vec{A} = \frac{1}{r}\hat{z}\left(\frac{B}{2}2r\right) = B\hat{z} \qquad (r>a)$$

$$\nabla\times\vec{A} = 0 \text{ for } (r<a)$$

Thus $\nabla\times\vec{A} = \vec{B}$ as desired