A nonrelativistic particle of mass m is bound in a central potential $V(\vec{r}) = V(r)$ that is finite at the origin.

(a) Show that its wave function in spherical coordinates can be written in the factorized form $\psi(\vec{r}) = R(r)\,Y(\theta,\phi)$. Show that the radial part T_r can be expressed in terms of the radial part $\vec{p}_r$ of the linear momentum operator $\vec{p}$, and that

$$\vec{p}_r \equiv \frac{1}{2}\left(\hat{e}_r\cdot\vec{p} + \vec{p}\cdot\hat{e}_r\right) = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r$$

→ (see Griffiths pgs 121-133)

In spherical coordinates, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right] + V\psi = E\psi$$

try solution $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\right)\right] + VRY = ERY$$

Divide by RY and multiply by $-\frac{2mr^2}{\hbar^2}$ to yield:

$$\underbrace{\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right)\right]}_{=\ell(\ell+1)} + \underbrace{\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]}_{=-\ell(\ell+1)} = 0$$

each term is brackets must be constant, let separation constant be $\ell(\ell+1)$

now verify that $\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right) \propto \vec{p}_r^{\,2} R$

$$p_r^2 R = -\hbar^2\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)\left(\frac{1}{r}\frac{\partial}{\partial r}(rR)\right) = -\hbar^2\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)\frac{1}{r}\left(R + r\frac{\partial R}{\partial r}\right)$$

$$= -\hbar^2\,\frac{1}{r}\frac{\partial}{\partial r}\left(R + r\frac{\partial R}{\partial r}\right) = -\frac{\hbar^2}{r}\left(\frac{\partial R}{\partial r} + \frac{\partial R}{\partial r} + r\frac{\partial^2 R}{\partial r^2}\right)$$

$$= -\hbar^2\left(\frac{2}{r}\frac{dR}{dr} + \frac{d^2R}{dr^2}\right) = -\frac{\hbar^2}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

thus, one can write $T_r = \frac{p_r^2}{2mr^2}$

b) Obtain the normalized angular function $Y(\theta,\phi)$ for S-states.
Show that the S-state radial function $R(r)$ could have solutions near the origin of the forms $R_1(r) = A$ and $R_2(r) = B/r$ where A, B are real constants. Show that both solutions are square-integrable near the origin.

→ for S-states, $\ell = 0$, and the angular equation becomes

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{d^2 Y}{d\phi^2} = 0$$

→ Y must be constant, normalization yields $\boxed{Y = \frac{1}{\sqrt{4\pi}}}$

for $R_1(r) = A$:

$$\frac{1}{A}\frac{d}{dr}\Big(r^2 \underbrace{\frac{dA}{dr}}_{=0}\Big) - \frac{2mr^2}{\hbar^2}(V(r) - E) = 0$$

$$-\frac{2mr^2}{\hbar^2}\left[V(r) - E\right] = 0$$

near origin: $\int_0^{R_0} A^2 r^2 dr = \frac{A^2 R_0^3}{3}$

where R_0 is small

for $R_2(r) = B/r$:

$$\underbrace{\frac{r}{B}\frac{d}{dr}\left(r^2 \frac{d}{dr}\left(\frac{B}{r}\right)\right)}_{=0} - \frac{2mr^2}{\hbar^2}(V(r) - E) = 0$$

$$-\frac{2mr^2}{\hbar^2}(V(r) - E) = 0$$

near origin: $\int_0^{R_0} \frac{B^2}{r^2} r^2 dr = B^2 R_0$

c) Show that $R_2(r)$ is not admissible as a solution for the wave function because the radial momentum $\vec{p}_r$ is not Hermetian for this type of wave function.

$$\int_0^\infty \frac{B}{r}\left(p_r\left(\frac{B}{r}\right)\right) r^2 dr = \int_0^\infty \frac{B}{r}\frac{\hbar}{i}\frac{1}{r}\Big(\underbrace{\frac{d}{dr}\left(\cancel{r}\frac{B}{\cancel{r}}\right)}_{=0}\Big) r^2 dr = 0$$

thus $\vec{p}_r$ is not Hermetian