A non-relativistic particle of mass m is bound in a central potential $V(\vec{r}) = V(r)$ that is finite at the origin.

(a) Show that its wave function in spherical coordinates can be written in the factorized form $\psi(\vec{r}) = R(r)\, Y(\theta,\phi)$ because the kinetic energy operator $T = T_r + T_L$ can be separated into a radial part and an angular part. Show that T_r can be expressed in terms of the radial part p_r of the linear momentum operator $\vec{p}$, and that

$$\vec{p}_r = \frac{1}{2}(\hat{r}\cdot\vec{p} + \vec{p}\cdot\hat{r}) = \frac{\hbar}{i}\,\frac{1}{r}\frac{\partial}{\partial r} r$$

where $\hat{r} = \frac{\vec{r}}{r}$

in gen. coord $\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$

spher. coord. $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

So,

$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

So, Schrödinger's eq is

$$-\frac{1}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$$

$$\Rightarrow \quad [\quad\quad] - 2m(V-E)\psi = 0$$

Now, let $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

substituting this expression for $\psi(r,\theta,\phi)$ into schrödinger's eq and multiplying through by $\frac{r^2}{R(r)Y(\theta,\phi)}$ yields

$$\underbrace{\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - 2mr^2(V-E)}_{=\ell(\ell+1)} + \underbrace{\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}}_{=-\ell(\ell+1)} =$$

so, we can separate schrödinger's eq into a radial part and an angular part. ✓

Now, we want to show that $p_r^2 R = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r}$ since $T_r = \frac{p_r^2}{2m}$

(and $T_L = \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta}$).

$$p_r^2 \rightarrow -\nabla^2 = \frac{-1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) = \frac{-1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

I'm not sure what we are supposed to do here really. So, I'll just show the useful relation

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(rA)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A}{\partial r} + A\right]$$

$$= \frac{1}{r}\frac{\partial A}{\partial r} + \frac{r}{r}\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} = \frac{2}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} \quad (1)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) = \frac{2r}{r^2}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} = \frac{2}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} \quad (2)$$

$$\therefore \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rA)}$$

(b) obtain the normalized angular function $Y(\theta,\phi)$ for S-states. Show that the S-state radial function $R(r)$ could have solutions near the origin of the forms

$$R_1(r) = A \quad \text{\&} \quad R_2(r) = B/r$$

Show that both solutions are square integrable near the origin.

we have

$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\Big|_{l=0 \text{ for s-states}} = 0$$

multiply by $Y\sin^2\theta$

$$\Rightarrow \quad \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = 0$$

this eq is only statisfied when $Y = \text{constant} \equiv C$. Normalization yields

$$1 = C^2\int_0^{2\pi}d\phi\int_0^{\pi}d\theta = C^2 4\pi$$

$$\therefore \quad \boxed{C = \frac{1}{\sqrt{4\pi}} = Y}$$

Now recall

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - 2mr^2(V-E) = l(l+1)\Big|_{l=0 \text{ for s-states}} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - 2mr^2(V-E)R = 0$$

if $R = A$, A = constant

$$0 - 2mr^2(V-E)R = 0$$

→ this problem is boring ...