Fall 1997 #14

A white dwarf star has a mass $M = 2\times10^{33}$ grams and a radius $R = 10^8$ cm. Assume that all electrons in the star are ionized and relativistic (so that $\epsilon(p) \approx pc$).

(a) Prove that the Fermi energy $\epsilon_f \propto \hbar c (N/V)^{1/3}$ with (N/V) the electron density. Show that the relativistic description is reasonable.

→ The volume of k space occupied is $\frac{4}{3}\pi k_f^3$
→ # of translational states per unit volume in k space $= \frac{V}{(2\pi)^3}$ (Reif 9.9.16)
→ degeneracy = 2 (electrons are fermions)

$$2\left(\frac{V}{(2\pi)^3}\right)\left(\frac{4}{3}\pi k_f^3\right) = N$$

$$k_f^3 = 3\pi^2\left(\frac{N}{V}\right)$$

now, use $\quad \epsilon_f = p_f c \quad \Rightarrow p_f = \frac{\epsilon_f}{c}$

$$\frac{p_f^2}{2m} = \frac{\hbar^2 k_f^2}{2m} \qquad \text{(Reif eq 9.9.4)}$$

$$\frac{\epsilon_f^2}{2mc^2} = \frac{\hbar^2 k_f^2}{2m} \quad \Rightarrow \quad k_f = \frac{\epsilon_f}{\hbar c}$$

$$\epsilon_f^3 = (\hbar c)^3 (3\pi^2)\left(\frac{N}{V}\right)$$

$$\boxed{\epsilon_f = \hbar c\,(3\pi^2)^{1/3}\left(\frac{N}{V}\right)^{1/3}}$$

assume all mass is helium, so $N = (2\times10^{33}\,\text{gm})\left(\frac{6.0\times10^{23}\,\text{atom}}{4\,\text{gm}}\right)\left(\frac{2\,e^-}{\text{atom}}\right)$

$N = 6\times10^{56}\,e^-$

$$\epsilon_f = (10^{-27}\,\text{erg}\cdot\text{sec})\left(\frac{1\,\text{eV}}{1.6\times10^{-12}\,\text{erg}}\right)\left(3\times10^{10}\,\frac{\text{cm}}{\text{s}}\right)(3\pi^2)^{1/3}\left(\frac{6\times10^{56}}{\frac{4}{3}\pi\,10^{24}}\right)^{1/3}$$

using a calculator yields $\quad \epsilon_f = 3\times10^6$ eV

the Fermi energy is much greater than the rest energy, so the relativistic approximation is appropriate

b) Within a star the reaction $e^- + p^+ \to n$ is possible the electron energy exceeds $\Delta\epsilon \simeq 0.8 \times 10^6$ eV. Compute the star's radius R_c for this to start to happen.

→ set $\epsilon_f = \Delta\epsilon$ and solve for R_c

$$\Delta\epsilon = \hbar c \, (3\pi^2)^{1/3} \left(\frac{N}{\frac{4}{3}\pi R_c^3}\right)^{1/3}$$

$$R_c = (3\pi^2)^{1/3} \hbar c \left(\frac{3N}{4\pi}\right)^{1/3} / \Delta\epsilon$$

$$R_c = (3\pi^2)^{1/3} (1.875 \times 10^{-5}) \left(\frac{3}{4\pi}\right)^{1/3} (6 \times 10^{56})^{1/3} / (0.8 \times 10^6)$$

$$\boxed{R_c = 3.8 \times 10^8 \text{ cm}}$$

c) What is the star's Fermi energy if $R = 0.1\, R_c$?

→ Assume that the star is entirely composed of neutrons, and that they are not relativistic.

$$k_f^3 = \frac{3\pi^2 N}{V} \quad ; \quad \text{use} \quad \epsilon_f = \frac{\hbar^2 k_f^2}{2m}$$

$$\left(\frac{2m\epsilon_f}{\hbar^2}\right)^{3/2} = \frac{3\pi^2 N}{V}$$

$$\epsilon_f = \left(\frac{\hbar^2}{2m}\right)\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

now $M = (2 \times 10^{33} \text{ gm})\left(\frac{1 \text{ neutron}}{1.6 \times 10^{-24} \text{ gm}}\right) = 1.25 \times 10^{57}$ neutrons $= N$

$R = 3.8 \times 10^7$ cm

$$\epsilon_f = \frac{(10^{-27})^2}{2(1.6 \times 10^{-24})} \left(\frac{3\pi^2 \cdot 1.25 \times 10^{57}}{\frac{4}{3}\pi (3.8 \times 10^7)^3}\right)^{2/3}$$

$$\boxed{\epsilon_f = 9.27 \times 10^{-8} \text{ eV}}$$