Fall 1997 # 14 (p 1 of 2)

A white dwarf star has a mass $M = 2 \times 10^{23}$ g and a radius $R = 10^{8}$ cm. You can assume that all the electrons in the star are ionized and relativistic (so $\epsilon(p) \approx pc$).

(a) Prove that the Fermi energy $\epsilon_F \propto \hbar c \left(\frac{N}{V}\right)^{1/3}$ with $\frac{N}{V}$ the electron density. Show that the relativistic description is reasonable.

From eq at bottom of p 390 Reif

$$N = 2 \frac{V}{(2\pi)^3} \left(\frac{4}{3}\pi k_F^3\right) \Rightarrow k_F = \left(3\pi^2 n\right)^{1/3}, \quad n = \frac{N}{V}$$

2: degeneracy

$\frac{V}{(2\pi)^3}$: # translational states per unit volume

$\frac{4}{3}\pi k_F^3$: volume of Fermi sphere

we are given $\epsilon_F \cong p_F c = \hbar k_F c = \hbar c \left(3\pi^2 n\right)^{1/3}$ (1)

Thus,

$$\boxed{\epsilon_F \propto \hbar c \, n^{1/3}}$$

Since N is so large, ϵ_F is greater than rest mass energy of electron.

To be precise, $N = [illegible] \times 10^{42}$ electrons $\Rightarrow \epsilon_F = 3 \times 10^{6}$ eV

Thus, relativistic description is reasonable.

(b) Within the star, a neutron can decay into a proton

$$n \rightarrow p^+ + e^-$$

with the electron energy $\Delta\epsilon \simeq 0.8 \times 10^{6}$ eV. The reverse reaction

$$e^- + p^+ \rightarrow n$$

is possible if the electron energy exceeds $\Delta\epsilon$. It is found that, as the radius R of the star decreases, the star's protons transform into neutrons. Explain and compute the star's radius R_c for this to start to happen.

) So, R_c corresponds to the value of R that allows for the energy given by eq 1 to equal $\Delta\epsilon$. That is

$$\epsilon = \Delta\epsilon = \hbar c \left(3\pi^2 \frac{N}{V}\right)^{1/3} \quad , V = \frac{4}{3}\pi R_c^3$$

$$\Rightarrow \Delta\epsilon = \hbar c \left(\frac{9\pi N}{4 R_c^3}\right)^{1/3}$$

$$\Rightarrow \boxed{R_c = \frac{\hbar c}{\Delta\epsilon} \left(\frac{9\pi N}{4}\right)^{1/3}}$$

putting in the #'s yields

$$\boxed{R_c = 3.8 \times 10^8 \text{ cm}}$$

)

(c) what is the star's Fermi energy if $R = 0.1 R_c$?

do not assume relativistic anymore. So

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

putting in the #'s yields

$$\boxed{\epsilon_F = 9.27 \times 10^{-8} eV}$$ ← justifies non-relativistic