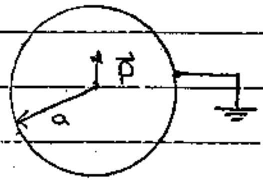
An ideal dipole $\vec{p}$ is located at the center of a hollow grounded conducting sphere of radius a.



a) Find the electrostatic potential Φ at an arbitrary point located within the sphere.

→ the potential of an ideal dipole is $V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{r^3}$

for $\vec{p} = p_0\hat{z}$:

$$V(r,\Theta) = \frac{p_0 \cos\Theta}{r^2}$$

The potential due to the sphere will have azimuthal symme and thus the general solution:

$$V_{sphere} = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\Theta) & r<a \\ \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\Theta) & r>a \end{cases}$$

by superposition:

$$V_{total} = \begin{cases} \sum_\ell A_\ell r^\ell P_\ell(\cos\Theta) + \frac{p_0\cos\Theta}{r^2} & r\leq a \\ \sum_\ell B_\ell r^{-(\ell+1)} P_\ell(\cos\Theta) + \frac{p_0\cos\Theta}{r^2} & r\geq a \end{cases}$$

find coefficients by using boundary condition $V(a) = 0$

for $r\leq a$:

$$\sum_\ell A_\ell a^\ell P_\ell(\cos\Theta) + \frac{p_0\cos\Theta}{a^2} = 0$$

can only be satisfied for $\ell = 1 \Rightarrow P_\ell(\cos\Theta) = \cos\Theta$

so

$$A_1 a\cos\Theta + \frac{p_0\cos\Theta}{a^2} = 0$$

$$A_1 = -p_0/a^3$$

so the potential inside the sphere is

$$\boxed{V(r,\Theta) = \left(-\frac{p_0}{a^3} r + p_0/r^2\right)\cos\Theta} \qquad r<a$$

outside the sphere:

$$B_1 a^{-2} \cos\Theta + \frac{p_0 \cos\Theta}{a^2} = 0 \quad \Rightarrow B_1 = -p_0$$

$$V(r,\Theta) = 0 \quad \text{outside sphere}$$

b) Deduce what is the energy of interaction between the ideal dipole and the grounded sphere.

→ Energy of interaction = (Energy of final system) − (Energy of isolated dipole)

energy of dipole in Electric field: $W = \vec{p} \cdot \vec{E}$

The isolated dipole has zero interaction energy, so the energy of interaction with the sphere is given by:

$$W = \vec{p} \cdot \vec{E}_{sphere}$$

E due to sphere is:

$$\vec{E} = -\nabla\left(\frac{-p_0 r \cos\Theta}{a^3}\right) = \frac{p_0 \cos\Theta}{a^3}\hat{r} - \frac{p_0 \sin\Theta}{a^3}\hat{\Theta}$$

recall $\hat{z} = \cos\Theta\,\hat{r} - \sin\Theta\,\hat{\Theta}$

$$\vec{E} = \frac{p_0}{a^3}\hat{z}$$

so $W = \vec{p} \cdot \vec{E} = \frac{p_0^2}{a^3}(\hat{z}\cdot\hat{z})$

$$\boxed{W = \frac{p_0^2}{a^3}}$$