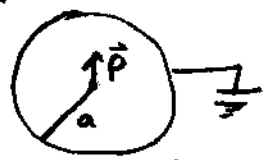
Fall 1997 #3 (p 1 of 2)

An ideal electric dipole $\vec{p}$ is located at the center of a hollow sphere of radius a, as in the figure below. The sphere is grounded.



(a) Find the electrostatic potential Φ at an arbitrary point located within the sphere.

The potential inside the spherical shell is given by the particular solution plus the homogeneous solution,

$$\left.\begin{aligned} \Phi_{particular} &= \frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{pr\cos\theta}{r^3} = \frac{p}{r^2}\cos\theta \\ \Phi_{homogeneous} &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \end{aligned}\right\} \quad r<a$$

So,

$$\Phi(r,\theta) = \frac{p}{r^2}\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

B.C.

$$\Phi(r=a,\theta) = 0 = \left[\frac{p}{r^2}\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\right]_{r=a}$$

$$\Rightarrow \quad \frac{p}{a^2}\cos\theta = -\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

So, only $l=1$ term survives the sum.

$$\frac{p}{a^2}\cos\theta = -A_1 a\cos\theta$$

$$\therefore \; A_1 = -\frac{p}{a^3}$$

Thus,

$$\boxed{\Phi(r,\theta) = \frac{p\cos\theta}{r^2} - \frac{p}{a^3} r\cos\theta = p\cos\theta\left[\frac{1}{r^2} - \frac{r}{a^3}\right]}$$

(b) Deduce what is the energy of interaction between the ideal dipole and the grounded sphere.

the energy of an ideal dipole $\vec{p}$ in an electric field $\vec{E}$ is given by

$$U = -\vec{p} \cdot \vec{E}$$

not from interaction!

$$\vec{E} = -\nabla \Phi = -\frac{\partial \Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\theta} = -\hat{r}\left[\cancel{\frac{2p\cos\theta}{r^3}} - \frac{p\cos\theta}{a^3}\right] + \frac{\hat{\theta}}{r}p\sin\theta\left(\cancel{\frac{1}{r^2}} - \frac{r}{a^3}\right)$$

$$= \hat{r}\frac{p\cos\theta}{a^3} - \hat{\theta}\frac{p\sin\theta}{a^3} = \frac{p}{a^3}\left[\hat{r}\cos\theta - \sin\theta\,\hat{\theta}\right]$$

$$\therefore \boxed{\vec{E} = \frac{p}{a^3}\hat{z}}$$

Thus,

$$u = -(p\hat{z}) \cdot \left(\frac{p}{a^3}\hat{z}\right)$$

$$\therefore \boxed{U = -\frac{p^2}{a^3}}$$