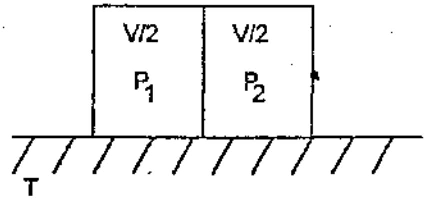
Fall 1997 #6

p 115

Consider an ideal gas of particles having mass m and initially confined to two isolated cells, each of volume V/2, as shown below. The cells are in contact with a heat reservoir at temperature T.



At time t=0 the left cell has pressure $p_1(0)$ and the right one $p_2(0)$. At this time a very small hole of area A is opened in the partition that separates the two cells and mixing occurs.

a) Calculate the pressure p_1 inside the left cell at an arbitrary time t > 0.

This is a problem dealing with effusion. The relevant equation gives the number of molecules which strike a unit area of a wall per unit time (Φ_0).

$$\Phi_0 = \frac{\overline{p}}{\sqrt{2\pi mkT}} \qquad \text{(Reif eq 7.11.13)}$$

One can use this information to write a differential equation for the pressure, which will yield $p_1(t)$ when solved.

Using the equation for Φ_0, write the change in the number of molecules in the left cell (Δn_1) as:

$$\Delta n_1 = \Phi_{right->left} - \Phi_{left->right}$$

$$\Delta n_1 = \frac{\overline{p_2} - \overline{p_1}}{\sqrt{2\pi mkT}} A\Delta t$$

Where Δt=time interval

If a sufficiently small hole is made in the wall of a container, the equilibrium of the gas inside is disturbed to a negligible extent (Reif pg 273). Thus, one can write an expression for the change in pressure by using the ideal gas equation of state:

$$\Delta p \frac{V}{2} = (\Delta n)kT$$

$$\Delta p = \frac{2kT\Delta n}{V}$$

$$\Delta p_1 = \frac{\overline{p_2} - \overline{p_1}}{\sqrt{2\pi mkT}}\left(\frac{2AkT}{V}\right)\Delta t$$

$$\frac{dp_1}{dt} = \lim_{\Delta t \to 0}\left(\overline{p_2} - \overline{p_1}\right)\frac{2A}{V}\sqrt{\frac{kT}{2\pi m}}$$

Now one can use the initial condition to write $\overline{p_2}$ in terms of $\overline{p_1}$. Since the total number of molecules must be conserved, one can write:

$$p_1\frac{V}{2} + p_2\frac{V}{2} = (n_1 + n_2)kT = NkT$$

Where N=total number of molecules

Solving for $\overline{p_2}$ yields:

$$\overline{p_2} = \frac{2NkT}{V} - \overline{p_1}$$

This expression for $\overline{p_2}$ holds at all times, allowing one to write

$$\frac{dp_1}{dt} = \left(\frac{2NkT}{V} - 2\overline{p_1}\right)\frac{2A}{V}\sqrt{\frac{kT}{2\pi m}} = \left(p_1 - \frac{NkT}{V}\right)\left(-\frac{4A}{V}\sqrt{\frac{kT}{2\pi m}}\right)$$

$$\frac{dp_1}{\left(p_1 - \frac{NkT}{V}\right)} = \frac{-4A}{V}\sqrt{\frac{kT}{2\pi m}}dt$$

$$\ln\left(p_1 - \frac{NkT}{V}\right) = \frac{-4At}{V}\sqrt{\frac{kT}{2\pi m}} + C$$

$$p_1(t) = \frac{NkT}{V} + C\exp\left(\frac{-4At}{V}\sqrt{\frac{kT}{2\pi m}}\right)$$

Use the initial condition to find C.

$$p_1(0) = \frac{NkT}{V} + C$$

$$C = p_1(0) - \frac{NkT}{V}$$

Thus,

$$p_1(t) = \frac{NkT}{V} + \left(p_1(0) - \frac{NkT}{V}\right)\exp\left(\frac{-4At}{V}\sqrt{\frac{kT}{2\pi m}}\right)$$

b) What should be the value of p_1 as $t \to \infty$?

Since the volumes of the two cells are equal, one would expect them to each contain half of the gas molecules for sufficiently large times. So one would expect to find that

$$p_1(\infty)\frac{V}{2} = \frac{NkT}{2}$$

$$p_1(\infty) = \frac{NkT}{V}$$

Evaluating the result of (a) for $t = \infty$ gives exactly this result.