Fall 1997 #8 p119

A two-dimensional oscillator has the Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(1+\delta xy)(x^2+y^2)$$

where $\hbar = 1$ and $\delta \ll 1$.

a) Give the wave functions for the two lowest energies when $\delta = 0$.

→ Write $H = H_x + H_y$

where $H_x = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2$, and similarly for H_y

Let $U(x,y) = u_1(x)\, u_2(y)$

where $u_1(x)$ is a solution of $H_x u_1(x) = E_1 u_1(x)$, etc

then $H\, U(x,y) = E\, U(x,y) = (E_1 + E_2)\, U(x,y)$

so the energy is equal to that of two independent oscillators.

The lowest energy is thus $2\cdot(\frac{1}{2}\hbar\omega) = \omega$ $(\hbar = 1)$

with wave function given by the product of two ground state harmonic oscillators. Recall $\psi_0(x) = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{\omega}{2}x^2}$ (Abers pg 6

Thus for $E_0 = \omega$

$$\boxed{\psi(x,y) = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}(x^2+y^2)}}$$

The next energy higher energy is when either of the oscillators is in its 1st excited state so $E_1 = \frac{1}{2}\omega + \frac{3}{2}\omega = 2\omega$. This state is 2-fold degenerate. Recall the wave function for the 1st excited state of an oscillator is $\psi_1(x) = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} x\, e^{-m\omega x^2/2\hbar}$

Thus:

$$\boxed{\psi_1(x,y) = \begin{cases} \sqrt{2}\,\frac{\omega^{3/4}}{\sqrt{\pi}}\, x\, e^{-\frac{\omega}{2}(x^2+y^2)} \\ \sqrt{2}\,\frac{\omega^{3/4}}{\sqrt{\pi}}\, y\, e^{-\frac{\omega}{2}(x^2+y^2)} \end{cases}} \Leftarrow \text{for } E = 2\omega$$

b) Evaluate the change in the energy levels for $\delta \neq 0$ to lowest order in δ.

→ ground state is non-degenerate, thus the 1st order shift in energy is given by $E^{(1)} = \langle\psi|H'|\psi\rangle$

where $H' = \delta(x^3 y + x y^3)$

Thus

$$E^{(1)} = \frac{\omega}{\pi}\int_{-\infty}^{\infty} x^3 e^{-\omega x^2} dx \int_{-\infty}^{\infty} y e^{-\omega y^2} dy + \frac{\omega}{\pi}\int_{-\infty}^{\infty} x e^{-\omega x^2} dx \int_{-\infty}^{\infty} y^3 e^{-\omega y^2} dy$$

Note that each of the four integrals is an integral of an odd function and thus equal to zero. Thus

$$\boxed{\text{There is no shift of the lowest state to 1st order.} \quad E^{(1)} = 0}$$

The first excited state is degenerate, so we need degenerate perturbation theory (see Griffiths pgs 227-234)

→ need to find matrix elements W_{ab}

$$W_{11} = \frac{2\omega^{3/2}}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x^3y+y^3x)\, x^2 e^{-\omega(x^2+y^2)}\,dx\,dy$$

$W_{11} = 0$ since integrals are odd

$W_{22} = 0$ since integrals are odd

$$W_{12} = \frac{2\omega^{3/2}}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x^3y+y^3x)\, x y\, e^{-\omega(x^2+y^2)}\,dx\,dy$$

$$= \frac{2\omega^{3/2}}{\pi}\delta\Big[\underbrace{\int_{-\infty}^{\infty} x^4 e^{-\omega x^2}dx}_{\frac{3}{4}\frac{\sqrt{\pi}}{\omega^{5/2}}}\underbrace{\int_{-\infty}^{\infty} y^2 e^{-\omega y^2}dy}_{\frac{1}{2}\frac{\sqrt{\pi}}{\omega^{3/2}}} + \underbrace{\int_{-\infty}^{\infty} y^4 e^{-\omega y^2}dy}_{\frac{3}{4}\frac{\sqrt{\pi}}{\omega^{5/2}}}\underbrace{\int_{-\infty}^{\infty} x^2 e^{-\omega x^2}dx}_{\frac{1}{2}\frac{\sqrt{\pi}}{\omega^{3/2}}}\Big]$$

(Reif A.4)

$$W_{12} = W_{21} = \frac{2\delta\omega^{3/2}}{\pi}\left(\frac{3\pi}{8\omega^4} + \frac{3\pi}{8\omega^4}\right) = \frac{3}{2}\frac{\delta}{\omega^{5/2}}$$

thus, one can write $\quad W = \begin{pmatrix} 0 & \frac{3}{2}\frac{\delta}{\omega^{5/2}} \\ \frac{3}{2}\delta/\omega^{5/2} & 0 \end{pmatrix}$

Solving for eigenvalues yields

$$\det\begin{pmatrix} -\lambda & \frac{3}{2}\delta/\omega^{5/2} \\ \frac{3}{2}\delta/\omega^{5/2} & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \lambda^2 - \left(\frac{3}{2}\frac{\delta}{\omega^{5/2}}\right)^2 = 0$$

$$\lambda = \pm\frac{3}{2}\frac{\delta}{\omega^{5/2}}$$

So the energy shifts of the 1st excited state are

$$\boxed{\Delta E = \pm\frac{3\delta}{2\omega^{5/2}}}$$