A 2-D oscillator has the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(1+\delta xy)(x^2+y^2)$$

where $\hbar = 1$ and $\delta \ll 1$. (see Yung-Kuo Lim QM #5048)

(a) give the wave functions for the two lowest energies when $\delta = 0$.

when $\delta = 0$, we have

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{\omega^2}{2}(x^2+y^2)$$

consider 1-D for now.

$$H = \frac{p_x^2}{2} + \frac{\omega^2}{2}x^2 \quad , \quad x = \frac{1}{\sqrt{2m\omega}}(a+a^+) \quad ; \quad a = \frac{1}{\sqrt{2\omega}}(\omega x + ip)$$

so, we know that the ground state is defined by (see Abers p73)

$$a|\psi_0\rangle = 0$$

$$\Rightarrow \langle x'|(\omega x + ip)|\psi_0\rangle = 0$$

for some reason you leave off the $\frac{1}{\sqrt{2\omega}}$ here?

$$\Rightarrow \omega\langle x'|x|\psi_0\rangle = -i\langle x'|p|\psi_0\rangle$$

$$\Rightarrow \omega x'\psi_0(x') = -\frac{d}{dx'}\psi_0(x') \Rightarrow -\omega x' dx' = \frac{d\psi_0(x')}{\psi_0(x')}$$

Solving this D.E. yields (dropping the primes)

$$\psi_0(x) = \left[\frac{\omega}{\pi}\right]^{1/4} e^{-\frac{\omega x^2}{2}}$$

where the coefficient comes from the normalization condition $1 = \int_{-\infty}^{\infty}|\psi_0(x)|^2 dx$

so, for two dimensions, we have

$$\psi_{00}(x,y) = \psi_0(x)\,\psi_0(y)$$

$$\therefore \boxed{\psi_{00}(x,y) = \sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega}{2}(x^2+y^2)}}$$

& $E_{00} = \omega(0+\frac{1}{2}) + \omega(0+\frac{1}{2})$

$\therefore E_{00} = \omega$

the next lowest energy wave function can be found by $a^\dagger$ on $\psi_0(x,y)$. That is,
consider 1-D first

$$|\psi_1\rangle = a^\dagger|\psi_0\rangle = \frac{1}{\sqrt{2\omega}}(\omega x - ip)|\psi_0\rangle$$

$$\Rightarrow \quad \psi_1(x) = \frac{1}{\sqrt{2\omega}}\left(\omega x \psi_0(x) - \frac{d\psi_0(x)}{dx}\right) =$$

$$= \frac{1}{\sqrt{2\omega}}\left[\omega x\left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{\omega}{2}x^2} + \left(\frac{\omega}{\pi}\right)^{1/4}\frac{2x\omega}{2} e^{-\frac{\omega}{2}x^2}\right]$$

$$= \frac{1}{\sqrt{2\omega}}\left(\frac{\omega}{\pi}\right)^{1/4} \omega x 2 e^{-\frac{\omega}{2}x^2}$$

$$\Rightarrow \quad \psi_1(x) = \sqrt{2\omega}\left(\frac{\omega}{\pi}\right)^{1/4} x e^{-\omega/2\, x^2}$$

So, in 2-D, we have

$$\psi_{01}(x,y) = \boxed{\psi_{10}(x,y) = 2\omega xy\left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}(x^2+y^2)}} \qquad \Leftarrow \; E = \omega(0+\tfrac{1}{2}) + \omega(1+\tfrac{1}{2})$$

$$E_{01} = 2\omega$$

(b) Evaluate the change in the energy levels for $\delta \neq 0$ (to lowest order in δ)

once again, just consider 2-D now (see Zettili p 473)

note;

$$E_{nn}^{(1)} = \langle \psi_n | \frac{\omega^2}{2}\delta xy(x^2+y^2)|\psi_n\rangle = 0$$

note that the integral over dx is an odd function $(x^3 + xy^2)$
and the same for the integral over dy.

So, since ψ_n must have same parity with ψ_n and since
the rest of the integrand is odd, these elements
vanish!

this is a degenerate perturbation problem. So, we need to creat a perturbation Hamiltonian matrix H_p.

$$H_p = \begin{pmatrix} \cancel{\langle \Psi_0 | H' | \Psi_0 \rangle}^{\,0} & \langle \Psi_0 | H' | \Psi_1 \rangle \\ \langle \Psi_1 | H' | \Psi_0 \rangle & \cancel{\langle \Psi_1 | H' | \Psi_1 \rangle}^{\,0} \end{pmatrix} - \begin{pmatrix} E^{(1)} & 0 \\ 0 & E^{(1)} \end{pmatrix}$$

$$\therefore \quad H_p = \begin{pmatrix} -E^{(1)} & \langle \Psi_0 | H' | \Psi_1 \rangle \\ \langle \Psi_1 | H' | \Psi_0 \rangle & -E^{(1)} \end{pmatrix}$$

where

$$\langle \Psi_0 | H' | \Psi_1 \rangle = \langle \Psi_1 | H' | \Psi_0 \rangle = \frac{\omega^2}{2}\delta \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\sqrt{\frac{\omega}{\pi}}\, e^{-\omega(x^2+y^2)}\right]\left[2\omega xy\left(\frac{\omega}{\pi}\right)^{1/2}\right] xy(x^2+y^2)$$

$$= \frac{\omega^2 \delta}{2}\, 2\omega\, \frac{\omega}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(x^2y^2x^2 + x^2y^2\,y^2\right) e^{-\omega(x^2+y^2)}$$

$$= \frac{\omega^4 \delta}{\pi} \int_{-\infty}^{\infty} dx\, e^{-\omega x^2} \int_{-\infty}^{\infty} dy\, \left[x^4y^2 + y^4x^2\right] e^{-\omega y^2}$$

note: $\int_{-\infty}^{\infty} e^{-\omega x^2} dx = \sqrt{\frac{\pi}{\omega}}$, $\int_{-\infty}^{\infty} x^2 e^{-\omega x^2} dx = \frac{\sqrt{\pi}}{2\omega^{3/2}}$, $\int_{-\infty}^{\infty} x^4 e^{-\omega x^2} dx = \frac{3\sqrt{\pi}}{4\omega^{5/2}}$

So,

$$\langle \Psi_0 | H' | \Psi_1 \rangle = \frac{\omega^4}{\pi}\delta \int_{-\infty}^{\infty} dx\, e^{-\omega x^2}\left[x^4 \frac{\sqrt{\pi}}{2\omega^{3/2}} + x^2 \frac{3\sqrt{\pi}}{4\omega^{5/2}}\right]$$

$$= \frac{\omega^4}{\pi}\delta\left[\frac{3\sqrt{\pi}}{4\omega^{5/2}}\frac{\sqrt{\pi}}{2\omega^{3/2}} + \frac{\sqrt{\pi}}{2\omega^{3/2}}\frac{3\sqrt{\pi}}{4\omega^{5/2}}\right]$$

$$= \frac{\omega^4}{\pi}\delta\left[\frac{3\pi}{8\omega^4} + \frac{3\pi}{8\omega^4}\right] = \delta\left[\frac{6}{8}\right] = \frac{3}{4}\delta$$

so,

$$\det\begin{pmatrix} -E^{(1)} & \frac{3}{4}\delta \\ \frac{3}{4}\delta & -E^{(1)} \end{pmatrix} = 0$$

$$\Rightarrow E^{(1)2} - \frac{9}{16}\delta^2 = 0$$

$$E^{(1)} = \pm\frac{3}{4}\delta$$

Thus, the corrected energies are

$$\boxed{\begin{aligned} E_{00} &= \omega \\ E_{01} = E_{10} &= 2\omega \pm \frac{3}{4}\delta \end{aligned}}$$

← no correction