Spring 1997 #1

A poorly conducting thin circular ring of electrical conductivity σ, mass m and radius r rotates with no friction about an axis perpendicular to a uniform magnetic field $\vec{B}$. It is set in motion with an initial frequency of rotation ω_o. Demonstrate that the frequency of rotation decays in time and calculate the corresponding decay constant τ.

→ We want to find the torque acting on the loop $\boxed{\vec{T} = \vec{m} \times \vec{B}}$ so we need to find the current in the loop. Use Faraday's Law to find EMF around the loop:

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$$

the flux through the loop can be described by $\Phi = BA\cos(\omega t)$ where $A = \pi r^2$ is the area of the loop

$$\mathcal{E} = +\frac{1}{c} B\pi r^2 \omega \sin(\omega t)$$

find the total resistance of the loop $\boxed{R = \rho\frac{2\pi r}{a} = \frac{2\pi r}{\sigma a}}$

where a is the cross-sectional area of the wire. Now the current can be found using $i = \mathcal{E}/R \Rightarrow I = \frac{B\pi r^2 \omega \sigma a}{c\, 2\pi r}\sin(\omega t)$

$$i = \frac{B r \omega \sigma a}{2c}\sin(\omega t)$$

The magnetic dipole moment is given by $\boxed{\vec{m} = i\vec{A}/c}$

$$\vec{m} = \frac{B\omega\sigma a \pi r^3}{2c^2}\sin(\omega t)\left[\cos(\omega t)\,\hat{z} + \sin(\omega t)\,\hat{y}\right]$$

Thus the torque is:

$$\vec{T} = \vec{m}\times\vec{B} = \left(\frac{B^2\omega\sigma a\pi r^3}{2c^2}\right)\sin(\omega t)\left[\cos(\omega t)\,(\hat{z}\times\hat{z}) + \sin(\omega t)\underbrace{(\hat{y}\times\hat{z})}_{=-\hat{x}}\right]$$

$$\vec{T} = \left(\frac{B^2\omega\sigma a\pi r^3}{2c^2}\right)\sin^2(\omega t)\,(-\hat{x})$$

⇒

by definition $\vec{T} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt}$ where $\vec{\alpha}$ is the angular acceleration and $\vec{\omega}$ is the angular velocity. Thus,

$$I\frac{d\vec{\omega}}{dt} = \frac{mr^2}{2}\frac{d\vec{\omega}}{dt} = \left(\frac{B^2\omega\sigma a\pi r^3}{2c^2}\right)\sin^2(\omega t)(-\hat{x})$$

$$\frac{d\omega}{dt} = \frac{-B^2\omega\sigma a\pi r}{mc^2}\sin^2(\omega t)$$

since ω is a function of time, we can find the average angular acceleration by taking time average of $\sin^2(\omega t) = \frac{1}{2}$

This gives:

$$\frac{d\omega}{dt} = \frac{-\omega B^2\sigma a\pi r}{2mc^2}$$

$$\frac{d\omega}{\omega} = \frac{-B^2\sigma a\pi r}{2mc^2}dt$$

$$\omega = \omega_0\exp\left(\frac{-B^2\sigma a\pi r}{2mc^2}t\right)$$

the general form of exponential decay is $\omega = \omega_0 e^{-t/\tau}$ for time constant τ. Comparison yields:

$$\boxed{\tau = \frac{2mc^2}{B^2\sigma a\pi r}}$$