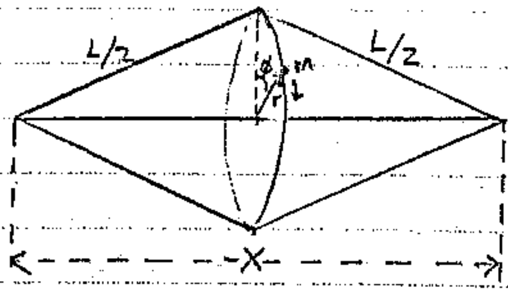
A weight of mass m is fixed to the middle point of a string of length L as shown in the figure. It rotates about an axis joining the string ends whose spacing is x. The system is in contact with its environment at a temperature T. (ignore gravity and assume that the string does not stretch or vibrate).



a) What is the Hamiltonian of this system in terms of the angle ϕ and the canonical momentum p_ϕ?

→ Let's start with the Lagrangian $L = T - V$ $\quad$ ($V=0$, ignoring gravity)

$$L = T = \frac{1}{2}(\cancel{m\dot{r}^2} + mr^2\dot{\phi}^2) = \frac{1}{2}mr^2\dot{\phi}^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$H = \left(\sum p_i \dot{q}_i\right) - L = m\left(r^2\dot{\phi}^2 - \frac{1}{2}r^2\dot{\phi}^2\right) = \frac{1}{2}mr^2\dot{\phi}^2 = \frac{p_\phi^2}{2mr^2}$$

$$\boxed{H = \frac{p_\phi^2}{2mr^2}}$$

b) Compute the thermal average $\langle p_\phi^2 \rangle$.

→ the partition function for the system is $Z = \int_{-\infty}^{\infty} e^{-\beta \frac{p_\phi^2}{2mr^2}} dp_\phi$

the average p_ϕ^2 is thus given by:

$$\langle p_\phi^2 \rangle = \frac{\int_{-\infty}^{+\infty} p_\phi^2 e^{-\beta \frac{p_\phi^2}{2mr^2}} dp_\phi}{\int_{-\infty}^{\infty} e^{-\beta \frac{p_\phi^2}{2mr^2}} dp_\phi} = -2mr^2 \frac{\partial}{\partial \beta}(\ln Z)$$

now calculate

$$Z = \int_{-\infty}^{\infty} e^{-\beta \frac{p_\phi^2}{2mr^2}} dp_\phi = \sqrt{\frac{2\pi m r^2}{\beta}} = \sqrt{2\pi m r^2}\, \beta^{-1/2}$$

$$\ln Z = \ln\left[(2\pi m r^2)^{1/2} \beta^{-1/2}\right] = \frac{1}{2}\ln(2\pi m r^2) - \frac{1}{2}\ln \beta$$

$$\frac{\partial}{\partial \beta}(\ln Z) = -\frac{1}{2}\beta^{-1} = -\frac{1}{2}kT$$

$\Rightarrow$

thus

$$\boxed{\langle p_\phi^2 \rangle = mr^2 kT}$$

c) Show that the tension $\vec{T}$ acting on the ends of the string required to keep x fixed is independent of m.

→ The radial component of the tension must equal the centrifugal force

$$T_r = \frac{mv^2}{r} = m\omega^2 r \qquad \text{(where } \omega = \text{angular velocity)}$$

now, $\langle p_\phi^2 \rangle = (m\omega)^2 \Rightarrow \omega^2 = \langle p_\phi^2 \rangle / m^2$

substituting for ω^2, $\langle p_\phi^2 \rangle$ yields:

$$\underset{\text{tension}}{T_r} = \frac{m(mr^2kT)}{m^2} = r^2 k \underset{\text{temp}}{T}$$

since the radial component of tension is mass independent, the total tension vector must be independent of m.