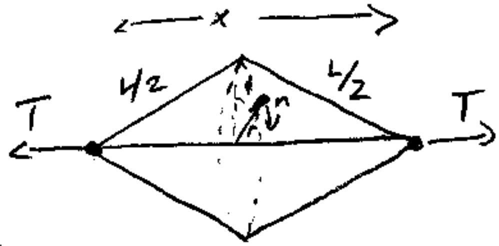
Spring 1997 #14

A weight of mass m is fixed to the middle point of a string of length L as shown in the figure. It rotates about an axis joining the string ends whose spacing is x. The system is in contact with its environment at a temperature T. Ignore gravity and assume string does not stretch or vibrate.



(r is the distance from m → center)

a) What is the Hamiltonian of this system in terms of the angle ϕ and the canonical momentum p_ϕ?

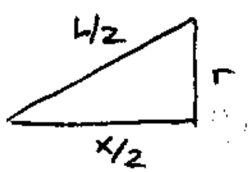
(See Kubo ch 2 #2)

The Hamiltonian is given by

$$H = \frac{p^2}{2m} = \cancelto{0}{\frac{p_r^2}{2m}} + \cancelto{0}{\frac{p_\theta^2}{2mr^2}} + \frac{p_\phi^2}{2mr^2\sin^2\theta}, \quad \theta = \frac{\pi}{2}$$

$$\Rightarrow \quad H = \frac{p_\phi^2}{2mr^2}$$

where the weight is at a distance



$$r^2 = +\frac{L^2}{4} - \frac{x^2}{4} = \frac{1}{4}(L^2 - x^2)$$

$$\Rightarrow \quad r = \frac{1}{2}\sqrt{+L^2 - x^2}$$

from the axis for all ϕ (since the triangle rotates around the circle)

Thus,

$$\boxed{H = \frac{2p_\phi^2}{m(L^2 - x^2)}}$$

(b) compute the thermal average $\langle P_\phi^2 \rangle$.

The partition function for this problem is given by

$$Z = \frac{L}{h}\int_{-\infty}^{\infty} e^{-\beta \epsilon}\, dp_\phi \quad \text{where} \quad \epsilon = \frac{2P_\phi^2}{m(L^2-x^2)} \equiv \alpha P_\phi^2$$

$$= \frac{L}{h}\int_{-\infty}^{\infty} e^{-\beta\alpha P_\phi^2}\, dp_\phi$$

So,

$$\langle P_\phi^2 \rangle = \frac{\frac{L}{h}\int_{-\infty}^{\infty} P_\phi^2\, e^{-\beta\alpha P_\phi^2}\, dp_\phi}{\frac{L}{h}\int_{-\infty}^{\infty} e^{-\beta\alpha P_\phi^2}\, dp_\phi} = \frac{\dfrac{\sqrt{\pi}}{2(\beta\alpha)^{3/2}}}{\dfrac{\sqrt{\pi}}{(\beta\alpha)^{1/2}}} = \frac{1}{2\beta\alpha}$$

$$\therefore \boxed{\langle P_\phi^2 \rangle = \frac{-1}{4} m(x^2-L^2)\, kT}$$

(c) Show that the tension T acting on the ends of the strings required to keep x fixed is independent of m.

The tension is defined as

$$T = -kT\, \frac{\partial \ln Z}{\partial x}$$

where

$$\ln Z = \ln\left(\frac{L}{h}\right) + \ln\left(\frac{\sqrt{\pi}}{(\beta\alpha)^{1/2}}\right) = \ln\left(\frac{L\sqrt{\pi}}{h\sqrt{\beta}}\right) - \frac{1}{2}\ln\alpha$$

So,

$$\frac{\partial \ln Z}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial x}\ln\alpha = \frac{-1}{2}\frac{\partial}{\partial x}\ln\left(\frac{2}{m(L^2-x^2)}\right) = +\frac{1}{2}\frac{\partial}{\partial x}\ln(x^2-L^2)$$

$$= +\frac{1}{2}\,\frac{2x}{x^2-L^2} = \frac{+x}{x^2-L^2}$$

Thus,

$$\boxed{T = \frac{-x\,KT}{x^2 - L^2}}$$

is independent of the mass.

alternative way to find tension:

the weight rotates with an angular velocity $\omega = \frac{d\phi}{dt}$ where one must balance the tension with the centrifugal force acting on it. That is,

$$T = \frac{m r \omega^2}{4r/x} = \frac{x P_\phi^2}{4 m r^4}$$

$$\Rightarrow T = \frac{2x}{L^2 - x^2} \frac{P_\phi^2}{2mr^2} = \frac{2x}{L^2 - x^2} \frac{1}{2} KT = \frac{x\,KT}{L^2 - x^2}$$