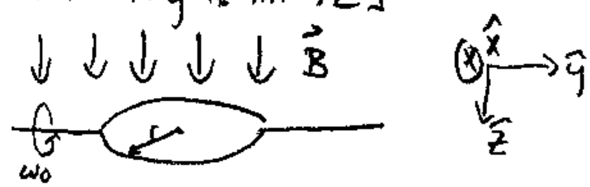
Spring 1997 #1 (p1 of 2)

A poorly conducting thin circular ring of electrical conductivity σ, and mass m, and radius r rotates with no friction about an axis perpendicular to a uniform magnetic field B. It is set in motion with an initial frequency ω_0. Demonstrate that the frequency of rotation decays in time and calculate the corresponding decay constant τ. [Hint: the moment of inertia of the ring is $mr^2/2$]



note:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \cancel{\vec{v} \times m\vec{v}}^{\,0} + \vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{N} \leftarrow \text{torque} \qquad (1)$$

we also know that the angular momentum of a rigid body is given by

$$\vec{L} = I\vec{\omega} \qquad , I \text{ is the moment of inertia} \qquad (2)$$

Thus, the torque on a rigid body is

$$\vec{N} = \frac{d I\vec{\omega}}{dt} = I\frac{d\vec{\omega}}{dt} \qquad , \text{since } I \text{ is constant} \qquad (3)$$

So, once we determine $\vec{N}$ for our system, we can get ω in the form of

$$\omega \propto e^{-t/\tau}$$

which will yield τ, the decay constant. So, that's the game to be played.

Now, let's find the torque, $\vec{N}$. The torque on a magnetic dipole is

$$\vec{N} = \vec{m} \times \vec{B}$$

where $\vec{B} = B\hat{z}$ and $\vec{m} = i\int d\vec{a}$, where this i is current. So, we have

$$\vec{N} = -mB\sin(\omega t)\,\hat{y} = -i\pi r^2 B\sin(\omega t)\,\hat{y} \qquad (4)$$

Now, we must find the current in terms of what we are given (see Griffiths 7.10)

$$i = \frac{\mathcal{E}}{R} = \frac{-\frac{d\Phi_B}{dt}}{\rho\frac{L}{A}} = \frac{-\frac{d}{dt}\int\vec{B}\cdot d\vec{a}}{\left(\frac{1}{\sigma}\right)\left(\frac{L}{A}\right)} = \frac{-\frac{d}{dt}B\pi r^2\cos(\omega t)}{2\pi r}\sigma A$$

So, we have

$$i = \frac{B\omega r \sin(\omega t)}{2} \sigma A \quad , A \text{ is cross-sectional area of wire}$$

substituting this expression for i into eq. (4), we get

$$\vec{N} = -\frac{B^2 \pi r^3 \omega \sigma A}{2} \sin^2(\omega t)\ \hat{y}$$

$$\Rightarrow \bar{\vec{N}} = -\frac{1}{4} B^2 \pi r^3 \omega \sigma A\ \hat{y}$$

substituting this expression into eq (3), we get

$$\frac{d\omega}{dt} = \frac{-1}{4I} B^2 \pi r^3 \omega \sigma A \quad , \text{ where } I = \frac{mr^2}{2}$$

$$= -\frac{B^2 \pi r \omega \sigma A}{2m}$$

$$\Rightarrow \frac{d\omega}{\omega} = -\frac{B^2 \pi r \sigma A}{2m} dt$$

$$\therefore \omega(t) = \omega_0 \exp\left[\frac{-B^2 \pi r \sigma A}{2m} t\right]$$

Thus,

$$\boxed{\tau = \frac{2m}{B^2 \pi r \sigma A}}$$