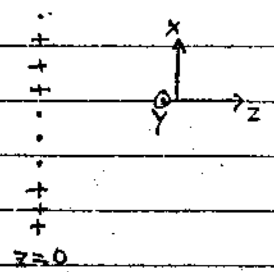
Spring 1997 #2

A two-dimensional free charge distribution is given by $\sigma_0 \delta(z) \cos kx$, where σ_0 and k are positive constants and $\delta(z)$ is a Dirac delta function. The variables x, y, z refer to the Cartesian coordinate system shown below:



The free charge is embedded inside a crystal dielectric whose dielectric tensor is uniaxial, ie. $\epsilon_{zz} = \epsilon_{\parallel}$, $\epsilon_{xx} = \epsilon_{yy} = \epsilon$ and all the off diagonal components are zero.

a) Write down the Poisson equation for the electrostatic potential ϕ for this system.

→ we know that $\vec{D} = \epsilon \vec{E} = \begin{pmatrix} \epsilon_{\perp} & 0 & 0 \\ 0 & \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

we also know that $\nabla \cdot \vec{D} = 4\pi \rho_{free} = 4\pi \sigma_0 \delta(z) \cos(kx)$

and furthermore $\vec{E} = -\nabla \phi$

→ so first, using $\vec{E} = -\nabla\phi$, we get

$$\vec{E} = -\frac{\partial}{\partial x}\phi\,\hat{x} - \frac{\partial}{\partial y}\phi\,\hat{y} - \frac{\partial}{\partial z}\phi\,\hat{z}$$

Next, use $\vec{D} = \epsilon \vec{E}$

$$\vec{D} = -\epsilon_{\perp}\frac{\partial}{\partial x}\phi\,\hat{x} - \epsilon_{\perp}\frac{\partial}{\partial y}\phi\,\hat{y} - \epsilon_{\parallel}\frac{\partial}{\partial z}\phi\,\hat{z}$$

Lastly use $\nabla \cdot \vec{D} = 4\pi \rho_{free}$

$$\nabla\cdot\vec{D} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(-\epsilon_{\perp}\frac{\partial}{\partial x}\phi\,\hat{x} - \epsilon_{\perp}\frac{\partial}{\partial y}\phi\,\hat{y} - \epsilon_{\parallel}\frac{\partial}{\partial z}\phi\,\hat{z}\right)$$

$$\nabla\cdot\vec{D} = -\epsilon_{\perp}\frac{\partial^2}{\partial x^2}\phi - \epsilon_{\perp}\frac{\partial^2}{\partial y^2}\phi - \epsilon_{\parallel}\frac{\partial^2}{\partial z^2}\phi = 4\pi\sigma_0\delta(z)\cos(kx)$$

Thus, the Poisson equation reads:

$$\boxed{\epsilon_{\perp}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \epsilon_{\parallel}\frac{\partial^2}{\partial z^2}\phi = -4\pi\sigma_0\delta(z)\cos(kx)}$$

b) Solve for the potential ϕ at a location $z > 0$

→ ϕ must not depend on y, $\phi = \phi(x, z)$

try solution of form $\phi(x,z) = X(x)Z(z)$ ⇒

for $z \neq 0$, the Poisson equation becomes:

$$\epsilon_\perp \left(\frac{d^2}{dx^2} X(x)\, Z(z)\right) + \epsilon_\parallel \left(\frac{d^2}{dz^2} X(x)\, Z(z)\right) = 0$$

divide by $X(x)\,Z(z)$ to get

$$\epsilon_\perp \frac{1}{X}\frac{d^2}{dx^2} X(x) + \epsilon_\parallel \frac{1}{Z}\frac{d^2}{dz^2} Z(z) = 0$$

each part must be constant, let $\frac{1}{X}\frac{d^2}{dx^2}X(x) = \frac{-\alpha^2}{\epsilon_\perp}$; $\frac{1}{Z}\frac{d^2}{dz^2}Z(z) = \frac{+\alpha^2}{\epsilon_\parallel}$

general solution

$$X(x) = A\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right) + \cancel{B\sin\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right)} \qquad (X(x) \text{ is even so } B=0)$$

$$Z(z) = C\exp\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}z\right) + E\exp\left(\frac{-\alpha}{\sqrt{\epsilon_\parallel}}z\right) \qquad ((C=0, \text{ for } z>0)\ (E=0,\ z<0))$$

for $z>0$ $\quad \phi = A\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right) E\exp\left(\frac{-\alpha}{\sqrt{\epsilon_\parallel}}z\right)$

so $E(z>0) = -\nabla\phi = +A\frac{\alpha}{\sqrt{\epsilon_\perp}}\sin\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right)E\exp\left(\frac{-\alpha}{\sqrt{\epsilon_\parallel}}z\right)\hat{x} + A\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right)E\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}\right)\exp\left(\frac{-\alpha}{\sqrt{\epsilon_\parallel}}\right.$

for $z<0$ $\quad \phi = A\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right) C\exp\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}z\right)$

so

$$E(z<0) = A\frac{\alpha}{\sqrt{\epsilon_\perp}}\sin\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right)C\exp\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}z\right)\hat{x} - A\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right)C\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}\right)\exp\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}z\right)$$

Now apply boundary conditions

$E_\parallel$ continuous yields $C = E$

$D_\perp$ discontinuous by σ_{free} yields (redefine $AE = F$)

$$F\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}\right)\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right) + F\left(\frac{\alpha}{\sqrt{\epsilon_\parallel}}\right)\cos\left(\frac{\alpha}{\sqrt{\epsilon_\perp}}x\right) = 4\pi\sigma_0\cos(kx)$$

comparison shows that $F = \frac{4\pi\sigma_0}{2\alpha}\sqrt{\epsilon_\parallel}$ and $\alpha = k\sqrt{\epsilon_\perp}$

thus, we find the potential for $z>0$

$$\boxed{\phi(x,z) = \frac{2\pi\sigma_0}{k}\sqrt{\frac{\epsilon_\perp}{\epsilon_\parallel}}\cos(kx)\exp\left(-\sqrt{\frac{\epsilon_\perp}{\epsilon_\parallel}}kz\right)}$$