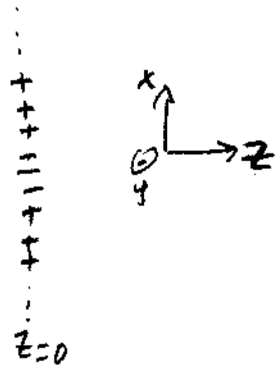
A 2-D Free charge distribution is given by $\sigma_0 \delta(z)\cos(kx)$, $\sigma_0, k > 0$; constants.



The free charge is embedded inside a crystal dielectric whose dielectric tensor is uniaxial, i.e., $\epsilon_{zz} = \epsilon_{\parallel}$, $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{\perp}$ and all the off-diagonal components are zero

(a) write down the Poisson equation for the electrostatic potential ϕ for this system. Use coordinate system shown to label derivatives properly. $\vec{E} = -\nabla\phi$

we know that

$$\vec{D} = \overleftrightarrow{\epsilon}\cdot\vec{E} = \begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \epsilon_\perp E_x \\ \epsilon_\perp E_y \\ \epsilon_\parallel E_z \end{pmatrix} = \begin{pmatrix} -\epsilon_\perp \frac{\partial\phi}{\partial x} \\ -\epsilon_\perp \frac{\partial\phi}{\partial y} \\ -\epsilon_\parallel \frac{\partial\phi}{\partial z} \end{pmatrix}$$

and

$$\nabla\cdot\vec{D} = 4\pi\rho_{free} = 4\pi\,\sigma_0\,\delta(z)\cos(kx)$$

combining these two expressions, we get

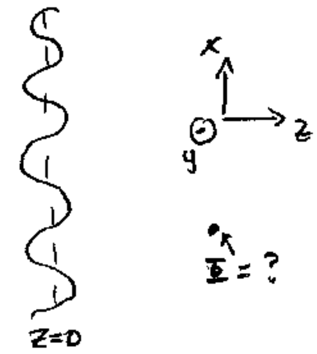
$$\nabla\cdot\begin{pmatrix} -\epsilon_\perp \frac{\partial\phi}{\partial x} \\ -\epsilon_\perp \frac{\partial\phi}{\partial x} \\ -\epsilon_\parallel \frac{\partial\phi}{\partial z} \end{pmatrix} = 4\pi\,\sigma_0\,\delta(z)\cos(kx)$$

$$\Rightarrow \epsilon_\perp \frac{\partial^2\phi}{\partial x^2} + \epsilon_\perp \frac{\partial^2\phi}{\partial y^2} + \epsilon_\parallel \frac{\partial^2\phi}{\partial z^2} = -4\pi\,\sigma_0\,\delta(z)\cos(kx)$$

$$\therefore \boxed{\epsilon_\perp\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \epsilon_\parallel \frac{\partial^2\phi}{\partial z^2} = -4\pi\,\sigma_0\,\delta(z)\cos(kx)}$$

(b) solve for the potential ϕ for a location $z>0$. Spring 1997 #2 (p 2 of 3)

(See Morales class notes 210A 11/16/04)



In this case, $\Phi(x,y,z) = \Phi(x,z)$, so solution is of the form $\Phi(x,z) = X(x)Z(z)$

substituting this into our result from part (a) and dividing through by $\frac{1}{XZ}$, we get

$$\frac{\epsilon_\perp}{X(x)}\frac{\partial^2}{\partial x^2}X(x) + \frac{\epsilon_\parallel}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = 0$$

only care about $z \neq 0$ now

the only way for this equation to be true is if each term is equal to a constant (since x & y are independent variables). So, we have

$$\left.\begin{aligned} \frac{\epsilon_\perp}{X(x)}\frac{\partial^2}{\partial x^2}X(x) &= -\alpha^2 \\ \frac{\epsilon_\parallel}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) &= \alpha^2 \end{aligned}\right\} \quad -\alpha^2 + \alpha^2 = 0 \ \checkmark$$

The solution to these are

$$X(x) = A\sin\left(\frac{\alpha}{n_\perp}x\right) + B\cos\left(\frac{\alpha}{n_\perp}x\right), \quad n_\perp = \sqrt{\epsilon_\perp}$$

$$Z(z) = C\exp\left(\frac{\alpha}{n_\parallel}z\right) + D\exp\left(\frac{-\alpha}{n_\parallel}z\right), \quad n_\parallel = \sqrt{\epsilon_\parallel}$$

For solution in the region $z>0$, $C=0$ (otherwise $\Phi \to \infty$ as $z \to \infty$). also, since the charge distribution is even, $A=0$. The general solution is then

$$\Phi(x,z) = \begin{cases} D'\cos\left(\frac{\alpha}{n_\perp}x\right)e^{-\frac{\alpha}{n_\parallel}z} & z \geq 0 \\ C'\cos\left(\frac{\alpha}{n_\perp}x\right)e^{\frac{\alpha}{n_\parallel}z} & z \leq 0 \end{cases}$$

since we know that the potential is continuous at $z=0$, we have that

$$D' = C'.$$

we also know that at $z=0$, $D^{\perp}_{z+} - D^{\perp}_{z^-} = 4\pi\sigma_{Free}$. That is,

$$\left(-\epsilon_{\parallel}\frac{\partial\Phi}{\partial z}\right)_{z=0^+} - \left(-\epsilon_{\parallel}\frac{\partial\Phi}{\partial z}\right)_{z=0^-} = 4\pi\sigma_0\cos(Kx)$$

$$\Rightarrow C'\frac{\alpha}{n_{\parallel}}\cos\left(\frac{\alpha}{n_\perp}x\right) + C'\frac{\alpha}{n_{\parallel}}\cos\left(\frac{\alpha}{n_\perp}x\right) = \frac{4\pi\sigma_0}{\epsilon_{\parallel}}\cos(Kx)$$

immediately we see that $\boxed{\alpha = Kn_\perp}$. Making this substitution we get

$$C'\left(\frac{Kn_\perp}{n_{\parallel}} + \frac{Kn_\perp}{n_{\parallel}}\right) = \frac{4\pi\sigma_0}{n_{\parallel}^2}$$

$$\Rightarrow C' = \frac{n_{\parallel}}{2Kn_\perp}\frac{4\pi\sigma_0}{n_{\parallel}^2} = \frac{2\pi\sigma_0}{Kn_\perp n_{\parallel}} = \boxed{\frac{2\pi\sigma_0}{K\sqrt{\epsilon_\perp\epsilon_{\parallel}}}}$$

So, the potential for a location $z>0$ is given by

$$\boxed{\Phi(x,z) = \frac{2\pi\sigma_0}{K\sqrt{\epsilon_\perp\epsilon_{\parallel}}}\cos(Kx)\,e^{-Kz\left(\frac{\epsilon_\perp}{\epsilon_{\parallel}}\right)^{1/2}}} \qquad z>0$$