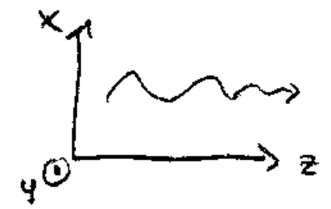
A spacecraft measures that the electric field of an electromagnetic wave propagating through the solar wind has the following functional dependence:

$$\vec{E}(z,t) = E_0 \, e^{-\alpha z} \cos(kz - \omega t)\,\hat{x}$$



where E_0, α, k, ω are real constants (positive).

(a) Deduce the oscillatory current density in this region of the solar wind.

From Ampère's law, we have

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{j}_F + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{j}_F = \frac{c}{4\pi}\left[\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right] \qquad (1)$$

where

$$\vec{D} = \epsilon \vec{E} = \epsilon E_0 e^{-\alpha z}\cos(kz - \omega t)\,\hat{x} \;\Rightarrow\; -\frac{1}{c}\frac{\partial \vec{D}}{\partial t} = -\frac{\omega}{c}\epsilon E_0 e^{-\alpha z}\sin(kz-\omega t)\,\hat{x} \quad (2$$

and $\hat{z} \times \hat{x} = \hat{y}$

$$\vec{B} = n\frac{(\vec{K}\times\vec{E})}{K} = n E_0 e^{-\alpha z}\cos(kz-\omega t)\,\hat{y} \;\Rightarrow\; \vec{H} = \frac{\vec{B}}{\mu} = \frac{n E_0}{\mu} e^{-\alpha z}\cos(kz-\omega t)\,\hat{y}$$

note: $n = \sqrt{\mu\epsilon}$, so, $\frac{n}{\mu} = \sqrt{\frac{\epsilon}{\mu}}$. also, $k = n\frac{\omega}{c}$

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so,

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & H_y(z) & 0 \end{vmatrix} = -\hat{x}\,E_0\sqrt{\frac{\epsilon}{\mu}}\,\frac{\partial}{\partial z}\left[e^{-\alpha z}\cos(kz-\omega t)\right]$$

$$= -\hat{x}\,E_0\sqrt{\frac{\epsilon}{\mu}}\left\{-\alpha e^{-\alpha z}\cos(kz-\omega t) - k e^{-\alpha z}\sin(kz-\omega t)\right\}$$

$$= \hat{x}\,E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\left\{\alpha\cos(kz-\omega t) + k\sin(kz-\omega t)\right\} \quad (3)$$

substituting eqs (3) and (2) into eq (1) yields

$$\vec{j}_F = \hat{x}\,\frac{c}{4\pi}\left[E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\Big(\alpha\cos(kz-\omega t)+k\sin(kz-\omega t)\Big) - \frac{\omega}{c}E_0\,e^{-\alpha z}\,\epsilon\sin(kz-\omega t)\right]$$

$$= \hat{x}\left\{\cos\gamma\left[\frac{\alpha c}{4\pi}E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\right] + \sin\gamma\left[\frac{c}{4\pi}E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\,k - \frac{c\,\omega}{4\pi c}\epsilon E_0 e^{-\alpha z}\right]\right\}$$

where $\gamma = kz - \omega t$

note in the second square brackets

$$[\ \ ] = \frac{cE_0}{4\pi}e^{-\alpha z}\left(\sqrt{\frac{\epsilon}{\mu}}\,k - \epsilon\frac{\omega}{c}\right) \underset{\frac{\omega}{c}=\frac{k}{n}}{=} \frac{cE_0}{4\pi}e^{-\alpha z}\underbrace{\left(\sqrt{\frac{\epsilon}{\mu}}\,k - \epsilon\frac{k}{n}\right)}_{=0 \text{ since } \frac{\epsilon}{n}=\frac{\epsilon}{\sqrt{\mu\epsilon}}=\sqrt{\frac{\epsilon}{\mu}}}$$

So, we have

$$\boxed{\vec{j}_F = \frac{c}{4\pi}\alpha E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\cos(kz-\omega t)\,\hat{x}}$$

(b) If the solar wind has mass density ρ, deduce the acceleration (time-averaged, per unit volume) being exerted by the wave on the solar wind at a location $z>0$.

The force per unit volume on the current density is given by

$$f = \frac{1}{c}\,\vec{J}\times\vec{B}$$

So,

$$f = \frac{1}{c}\,\frac{c}{4\pi}\alpha E_0\sqrt{\frac{\epsilon}{\mu}}\,e^{-\alpha z}\cos(kz-\omega t)\;nE_0e^{-\alpha z}\cos(kz-\omega t)\;\hat{x}\times\hat{y}$$

$$= \frac{\alpha}{4\pi}E_0^2\,n\sqrt{\frac{\epsilon}{\mu}}\,e^{-2\alpha z}\cos^2(kz-\omega t)\,\hat{z} \quad , \quad n\sqrt{\frac{\epsilon}{\mu}} = \epsilon$$

$$\Rightarrow \quad f = \frac{\alpha}{4\pi} E_0^2 \, \epsilon \, e^{-2\alpha z} \cos^2(kz - \omega t)\, \hat{z}$$

Then the time-averaged force per unit volume is

$$\boxed{\langle f \rangle = \frac{\alpha}{8\pi} \epsilon E_0^2 \, e^{-2\alpha z} \, \hat{z}}$$

Thus, the time-averaged, per unit volume acceleration is

$$\boxed{\langle a \rangle = \frac{\langle f \rangle}{\rho} = \frac{\alpha \epsilon}{8\pi\rho} E_0^2 \, e^{-2\alpha z} \, \hat{z}}$$

mass density of solar wind.