Spring 1997 #6

A system consisting of N atoms with spin 1/2 is placed in a magnetic field $\vec{H}$ and is kept at temperature T. Find the entropy, internal energy, total magnetic moment, and the specific heat of the system. (Assume the gyromagnetic ratio is $g=2$)

the energy of a single atom is $E = -\vec{\mu}\cdot\vec{H}$ (Ref eq. 7.8.1)
where $\vec{\mu} = g\mu_0 \vec{J}$, thus

$$E = -g\mu_0 H J_z$$ (Ref eq 7.8.3)

so $E = \pm\frac{1}{2} g\mu_0 H = \pm \mu_0 H$

the partition function for a single atom is therefore.

$$Z_1 = e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H} = 2\cosh(\beta\mu_0 H)$$

for N atoms, assumed distinguishable, the partition function is

$$Z = (2\cosh(\beta\mu_0 H))^N$$

$$F = -kT\ln Z = -kTN\ln\left[2\cosh(\beta\mu_0 H)\right]$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = +kN\ln\left[2\cosh(\beta\mu_0 H)\right] + \frac{kTN\,(2\sinh(\beta\mu_0 H))(\mu_0 H)}{2\cosh(\beta\mu_0 H)\;(-kT^2)}$$

$$\boxed{S = kN\ln\left[2\cosh\left(\frac{\mu_0 H}{kT}\right)\right] - \frac{N\mu_0 H}{T}\tanh\left(\frac{\mu_0 H}{kT}\right)}$$

$$E = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{-1}{(2\cosh(\beta\mu_0 H))^N}\left[N(2\cosh(\beta\mu_0 H))^{N-1}\right]2\sinh(\beta\mu_0 H)\mu_0$$

$$= \frac{-N\mu_0 H\sinh(\beta\mu_0 H)}{\cosh(\beta\mu_0 H)} = -N\mu_0 H\tanh(\beta\mu_0 H)$$

$$\boxed{E = -N\mu_0 H\tanh(\beta\mu_0 H)}$$

the total energy is given by $E = -MH$ so $M = -E/H$

$$\boxed{\overline{M} = N\mu_0\tanh(\beta\mu_0 H)}$$

Specific heat $C_v = \left(\frac{dE}{dT}\right)_v = \left(\frac{dE}{d\beta}\right)\left(\frac{d\beta}{dT}\right)$

$$C_v = -N\mu_0 H\left(\operatorname{sech}^2(\beta\mu_0 H)\right)\mu_0 H\left(\frac{-1}{kT^2}\right)$$

$$\boxed{C_v = \frac{N(\mu_0 H)^2}{kT^2}\operatorname{sech}^2\left(\frac{\mu_0 H}{kT}\right)}$$