A system consisting of N atoms with spin 1/2 is placed in a magnetic field H and is kept at temperature T. Find the entropy, the internal energy, the total magnetic moment, and the specific heat of the system. $(g=2)$

we know that the energy of a single atom is

$$E = -\vec{\mu}\cdot\vec{H} = \pm\mu_0 H \qquad , |\vec{\mu}| = \frac{g\mu_0}{2} \text{ for spin } 1/2 \text{ partic}$$

Assume the atoms are indistinguishable. So, we have for one atom

$$Z_1 = e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H}$$

Then for N indistinguishable atoms, we have

$$Z = \frac{1}{N!} Z_1^N = \frac{1}{N!}\left(e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H}\right)^N$$

Then the free energy is

$$F = -KT\ln Z = +KT\ln N! - KTN\ln\left(e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H}\right)$$

$N \gg 1$

$$\cong KTN\ln N - KTN - KTN\ln\left(e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H}\right)$$

$$= KTN\left[\ln N - 1 - \ln\left(e^{-\beta\mu_0 H} + e^{+\beta\mu_0 H}\right)\right]$$

(i) entropy (note: $e^{-\beta\mu_0 H} + e^{\beta\mu_0 H} = 2\cosh(\beta\mu_0 H)$)

So,

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -KN\left[\ln N - 1 - \ln\left[2\cosh(\beta\mu_0 H)\right]\right] +$$

$$+ KTN\frac{2\sinh(\beta\mu_0 H)}{2\cosh(\beta\mu_0 H)}\cdot\left(\frac{-\mu_0 H}{KT^2}\right)$$

$$\therefore \boxed{S = KN\left[-\ln N + 1 + \ln\left[2\cosh(\beta\mu_0 H)\right] - \frac{\mu_0 H}{KT}\tanh(\beta\mu_0 H)\right]}$$

(ii) internal energy

$$E = -\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}\left[N\ln N - N - N\ln\left(2\cosh(\beta\mu_0 H)\right)\right]$$

$$= -N\frac{2\sinh(\beta\mu_0 H)}{2\cosh(\beta\mu_0 H)}\cdot\mu_0 H$$

$$\therefore \boxed{E = -N\mu_0 H\tanh(\beta\mu_0 H)} = N E_{atom}$$

(iii) total magnetic moment

From Reif eq 7.8.20, we have an expression for the mean magnetic moment per unit volume

$$\bar{M}_z = N_0\bar{\mu}_z \quad , \text{where } N_0 = \frac{N}{V} \text{ \& } \bar{\mu}_z = \frac{-E_{atom}}{H}$$

So the total magnetic moment is

$$m = \frac{-N E_{atom}}{H} = \frac{N\mu_0 H\tanh(\beta\mu_0 H)}{H}$$

$$\therefore \boxed{m = N\mu_0\tanh(\beta\mu_0 H)}$$

(iv) specific heat capacity of system.

$$C_V = \frac{\partial E}{\partial\beta}\frac{\partial\beta}{\partial T} = -\frac{1}{kT^2}\frac{\partial E}{\partial\beta} = +\frac{1}{kT^2}N\mu_0 H\,\mathrm{sech}^2(\beta\mu_0 H)\ \mu_0 H$$

$$\therefore \boxed{C_V = \frac{N\mu_0^2 H^2}{kT^2}\mathrm{sech}^2(\beta\mu_0 H)}$$