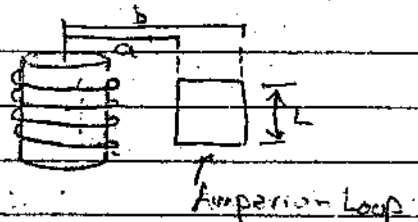
Spring 1997 #7

Consider a very long solenoid that is aligned with the z-axis of an ordinary Cartesian coordinate system. A constant electric current is passed through the wire.

a) Use what you know about magnetic field lines to give a plausibility argument why the field outside a very long solenoid is very weak.

(see Griffith's pgs 152-153)

Start by applying Ampere's Law to a loop drawn entirely outside a solenoid as shown below. For this loop:



$$\oint \vec{B}\cdot d\vec{l} = [B(a) - B(b)]\,L = \frac{4\pi}{c} I_{enc} = 0$$

so we see $B(a) = B(b)$. Thus, outside the solenoid the magnetic field is independent of the distance from the axis. However, we know that $\vec{B}$ must go to zero infinitely far away. so $\vec{B} = 0$ outside ideal solenoid, or $\vec{B}$ is very weak outside a very long real solenoid.

b) Consider the ideal limit of an infinitely long solenoid when the magnetic field $\vec{B}$ is vanishingly small outside the solenoid. The vector potential $\vec{A}$ is a solution of $\vec{B} = \nabla\times\vec{A}$. Thus $\nabla\times\vec{A} = 0$ outside of the solenoid. Show that curl $\vec{A}$ can satisfy this equation without $\vec{A}$ itself being zero: $\nabla\times\vec{A} = 0$ but $\vec{A} \neq 0$

→ outside the solenoid, let $\vec{A}$ equal the gradient of any scalar function. Since the curl of a gradient is always zero, this will always give $\nabla\times\vec{A} = 0$

c) Suppose that the total magnetic flux inside the solenoid is Φ. Prove that $\vec{A} \neq 0$.

(see Griffith's pg 230-231)

Notice that:

$$\oint \vec{A}\cdot d\vec{l} = \int(\nabla\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi$$

since Φ is nonzero, the line integral $\oint \vec{A}\cdot d\vec{l}$ is non-zero and hence $\vec{A} \neq 0$.

d) Find a vector potential which is consistent with (c).

$$\oint \vec{A}\cdot d\vec{l} = |A| 2\pi r$$

inside the solenoid $\Phi = B\pi r^2$

thus

$$|A| = \frac{B\cancel{\pi} r^{\cancel{2}}}{2\cancel{\pi}\cancel{r}} = \frac{Br}{2}$$

outside the solenoid $\Phi = B\pi R^2$ (R = radius of solenoid)

so

$$|A| = \frac{B\cancel{\pi}R^2}{2\cancel{\pi}r}$$

using the right hand rule, and assuming $\vec{B} = B\hat{z}$, we find that $\vec{A}$ points in the $\hat{\phi}$ direction, thus

$$\boxed{\vec{A} = \begin{cases} \frac{Br}{2}\hat{\phi} & r<R \\ \frac{BR^2}{2r}\hat{\phi} & r>R \end{cases}}$$