Consider a very long solenoid that is aligned with the z-axis of an ordinary, Cartesian coordinate system. A constant electric current is passed through the wire.

(a) Use what you know about magnetic field lines to give a plausibility argument why the field outside a very long solenoid is very weak.

(see example 5.9 Griffiths; p 227 - 229.)

radius of solenoid

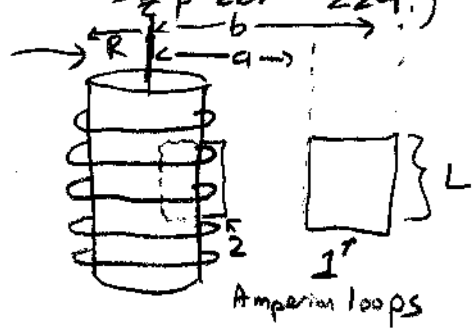


Figure 1:

The $\vec{B}$-field cannot have a radial component due to the direction of the current
The $\vec{B}$-field cannot have a circumferential component either b/c of the current's direction

That is,


$$\oint \vec{B} \cdot d\vec{l} = B_\phi (2\pi r) = \frac{4\pi}{c} I_{enc} = 0$$

← encloses no current

So, the magnetic field of an infinite, closely wound solenoid runs parallel to the axis.

Consider figure 1, loop 1

$$\oint \vec{B} \cdot d\vec{l} = [B(a) - B(b)]\, L = \frac{4\pi}{c} \cancel{I_{enc}}^{\,0} = 0$$

$$\Rightarrow \quad B(a) = B(b)$$

← Field outside does not depend on distance from axis. Since we require it to vanish for large r, there can't be a $\vec{B}$-field anywhere outside an infinitely-long solenoid.

From this, we can infer that the field outside a very long solenoid is either zero or very weak.

(b) consider the ideal limit of an infinitely long solenoid. So $B \to 0$ and since $\vec{B} = \nabla \times \vec{A}$, $\nabla \times \vec{A} = 0$. Show how $\nabla \times \vec{A}$ can equal zero without $\vec{A}$ being zero itself.

if $\vec{A} = \nabla f$, where f is a scalar function

then $\nabla \times \vec{A} = \nabla \times \nabla f = 0$

or

let $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z} = \vec{r} \quad \Rightarrow \nabla \times \vec{A} = 0$

since $$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \hat{x}\cdot 0 + \hat{y}\cdot 0 + \hat{z}\cdot 0 = 0$$

(c) let $\Phi_B$ be the total magnetic flux in the solenoid. Prove that $\vec{A} \neq 0$.

$$\Phi_B = \int \vec{B}\cdot d\vec{a} = \int (\nabla \times \vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$$

Thus, if $\Phi_B \neq 0 \Rightarrow \vec{A} \neq 0$ (as long as $d\vec{l} \not\perp \vec{A}$ which it is not)

(d) Find the vector potential which is consistent with part (c).

$$\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{a}$$

$$\Rightarrow |\vec{A}|\, 2\pi r = |\vec{B}|\, \pi r^2 \quad \Rightarrow \quad A = \frac{Br}{2} \qquad r < R$$

and

$$|\vec{A}|\, 2\pi r = |\vec{B}|\, \pi R^2 \quad \Rightarrow \quad A = \frac{BR^2}{2r} \qquad r > R$$

Thus,

$$\boxed{\vec{A} = \frac{B}{2}\hat{\phi}\begin{cases} r & r \le R \\ \dfrac{R^2}{r} & r \ge R \end{cases}}$$