# Spring 1997 #9

Consider a two-state quantum mechanical system with energies $E_1$ and $E_2$. Suppose it is acted on by a time independent perturbation represented by the matrix:

$$H' = \delta V = a\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

where $a$ is small compared to $E_1 - E_2$. If at time $t=0$, it is in state 1, find the probability that it is in state 2 at time T.

$\Rightarrow$ assume the eigenvectors of the unperturbed Hamiltonian are $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues $E_1$ and $E_2$ respectively

$|\Psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Baker-Hausdorf ?

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle = e^{-iH_0t/\hbar}e^{-iH't/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\Psi(t)\rangle = e^{-iE_1t/\hbar}e^{-iH't/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we need to express $|1\rangle$ in terms of the eigenstates of $H'$

find eigenvalues:

$$\det\begin{pmatrix} -\lambda & ia \\ -ia & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 + i^2a^2 = \lambda^2 - a^2 = 0$$

$$\lambda = \pm a$$

find $\lambda = +a$ eigenstate:

$$\begin{pmatrix} -a & ia \\ -ia & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{matrix} -\cancel{a}x + i\cancel{a}y = 0 \\ -i\cancel{a}x - \cancel{a}y = 0 \end{matrix} \Rightarrow y = -i$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad |+a\rangle \text{ eigenstate}$$

find $\lambda = -a$ eigenstate:

$$\begin{pmatrix} a & ia \\ -ia & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{matrix} \cancel{a}x + i\cancel{a}y = 0 \\ -i\cancel{a}x + \cancel{a}y = 0 \end{matrix} \Rightarrow y = ix$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad |-a\rangle \text{ eigenstate}$$

Thus $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[|+a\rangle + |-a\rangle\right]$

for later use note

$$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}\left[|+a\rangle - |-a\rangle\right]$$

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}\left[e^{-iat/\hbar}|+a\rangle + e^{+iat/\hbar}|-a\rangle\right]$$

The probability to find the particle in state $|2\rangle$ is given by $|\langle 2|\psi(t)\rangle|^2$

$$\langle 2|\psi(t)\rangle = \left[\frac{1}{\sqrt{2}}\langle +a| - \frac{i}{\sqrt{2}}\langle -a|\right] e^{-iE_1 t/\hbar}\left[e^{-iat/\hbar}|+a\rangle + e^{+iat/\hbar}|-a\rangle\right]$$

$$\langle 2|\psi(t)\rangle = e^{-iE_1 t/\hbar}\left(\frac{i}{\sqrt{2}}e^{-iat/\hbar} - \frac{i}{\sqrt{2}}e^{+iat/\hbar}\right)$$

$$|\langle 2|\psi(t)\rangle|^2 = e^{+iE_1 t/\hbar}\left(\frac{-i}{\sqrt{2}}e^{iat/\hbar} + \frac{i}{\sqrt{2}}e^{-iat/\hbar}\right) e^{-iE_1 t/\hbar}\left(\frac{i}{\sqrt{2}}e^{-iat/\hbar} - \frac{i}{\sqrt{2}}e^{ia}\right.$$

$$|\langle 2|\psi(t)\rangle|^2 = \frac{-i^2}{2} + \frac{i^2}{2}e^{2iat/\hbar} + \frac{i^2}{2}e^{-2iat/\hbar} - \frac{i^2}{2}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\left(\cos(2at/\hbar) + \cancel{i\sin(2at/\hbar)}\right) - \frac{1}{2}\left(\cos(2at/\hbar) - \cancel{i\sin(2a}\right.$$

$$\boxed{|\langle 2|\psi(t)\rangle|^2 = 1 - \cos(2at/\hbar)}$$